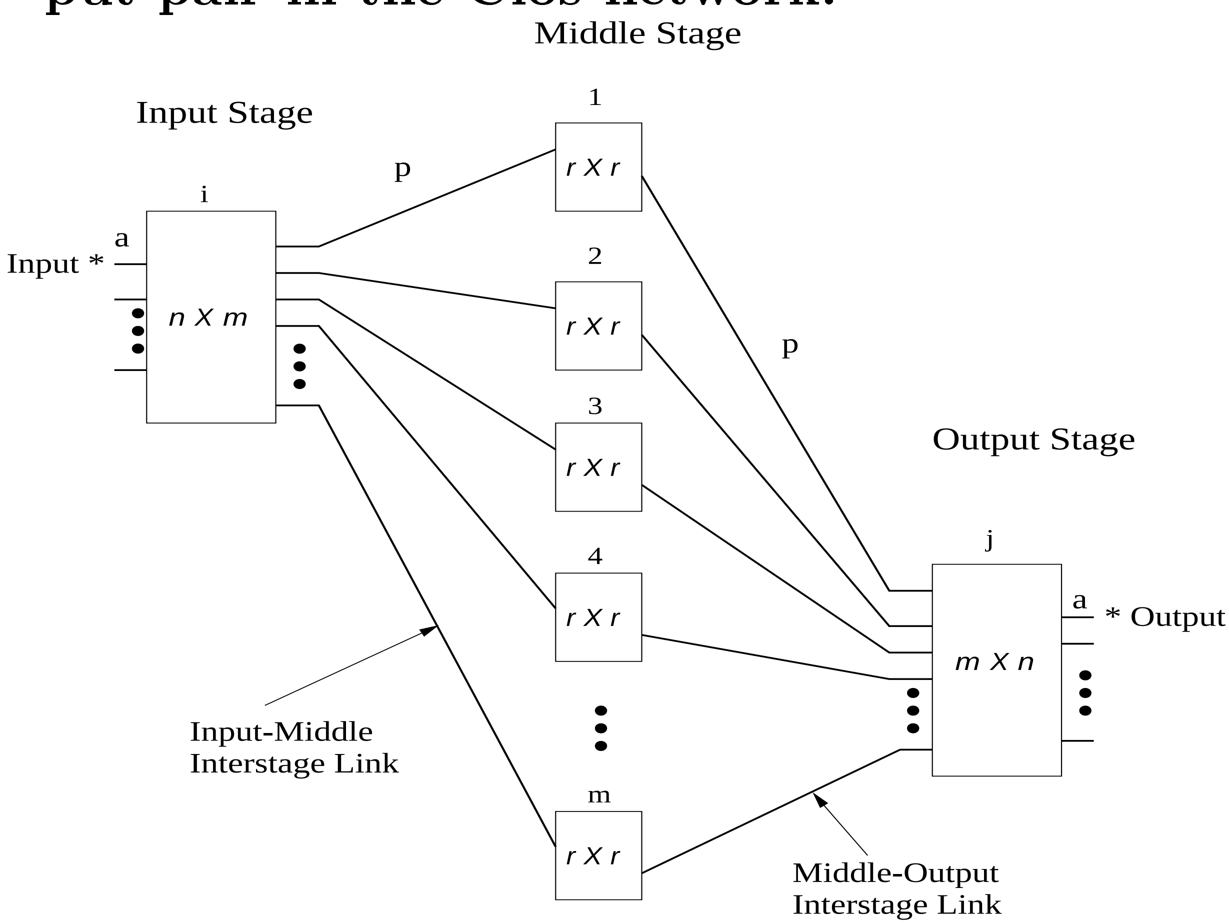
PERFORMANCE MODELING

Focus on analyzing the blocking behavior of a network that does not satisfy the nonblocking condition, that is, develop analytical models on blocking probability.

Lee's Model for unicast Clos networks:

- **The m paths between a given input and output pair in the Clos network:**



- a: **the probability that a typical input (or output) link is busy**
- p: **the probability that an interstage link is busy.**
- **Random routing strategy is used:**
  assume that the incoming traffic is uniformly distributed over the m **interstage links and the events that individual links in the network are busy are independent.**
- **The probability that an interstage link is busy is** $p = \frac{an}{m}$**.**
- **The probability that an interstage link is idle is** $q = 1 - p$**.**
- **The probability that a path (consisting of two interstage links) cannot be used for a connection is** $1 - q^2$**.**

- **The blocking probability (i.e., all m paths cannot be used)**

$$P_B = [1 - q^2]^m$$

- **Example:**

$$n = 32, m = 2n - 1 = 63, a = 1$$

$$P_B = 2.5 \times 10^{-8} \neq 0$$

Does not meet the deterministic nonblocking condition.

Jacobaeus' Model:

- **The blocking probability**

$$P_B = \frac{(n!)^2(2-a)^{2n-m}a^m}{m!(2n-m)!}$$

- **Example:**

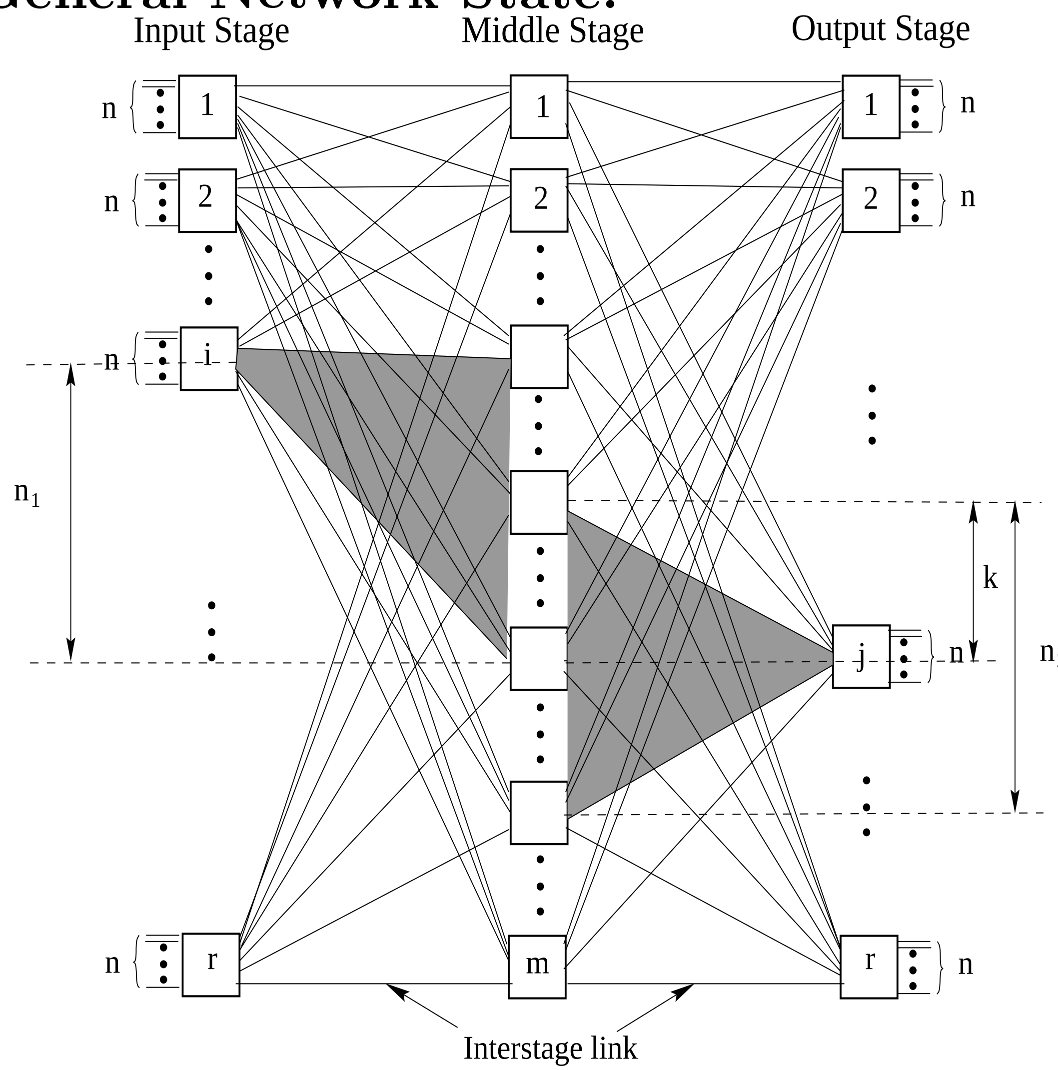
$$n = 32, m = 2n - 1 = 63, a = 1$$

$$P_B = 3.4 \times 10^{-20} \neq 0$$

Still does not meet the deterministic nonblocking condition.

The New Analytical Model

A General Network State:



A three-stage Clos network with n_1 busy input-middle interstage links from input stage switch i, n_2 busy middle-output interstage links to output stage switch j, and k pairs of the interstage links overlapped.

Notations and Assumptions

- **If a busy input-middle interstage link and a busy middle-output interstage link share the same middle stage switch, this pair of links is said to be *overlapped*.**
- **Let $\boldsymbol{n_1}$ denote the event that there are n_1 busy input-middle interstage links from input stage switch i.**
- **Let $\boldsymbol{n_2}$ denote the event that there are n_2 busy middle-output interstage links to output stage switch j.**
- **Random routing strategy is used: assume that the incoming traffic is uniformly distributed over the m interstage links and the events that individual links in the networks are busy are independent.**

Probability of k Interstage Links Overlapped

Lemma 1 ***Given events n_1 and n_2, the probability that k pairs of links are overlapped in the Clos network is given by***

$$\Pr\{k \textit{ pairs of links overlapped} \mid n_1, n_2\}$$
$$= \frac{\binom{n_1}{k}\binom{m-n_1}{n_2-k}}{\binom{m}{n_2}} = \frac{\binom{n_2}{k}\binom{m-n_2}{n_1-k}}{\binom{m}{n_1}}$$

Proof.

- $\binom{m}{n_1}\binom{m}{n_2}$ **ways to choose** n_1 **busy input-middle interstage links and** n_2 **busy middle-output interstage links.**

- k **pairs of overlapped links can be constructed as follows:**
  - $\binom{m}{n_1}$ **ways to choose** n_1 **busy input-middle interstage links;**
  - $\binom{n_1}{k}$ **ways to choose** k **input-middle interstage links overlapped with** k **middle-output interstage links;**
  - $\binom{m-n_1}{n_2-k}$ **ways to choose the rest of** $n_2 - k$ **middle-output interstage links.**

- **The probability that** k **pairs of links are overlapped is**

$$\frac{\binom{m}{n_1}\binom{n_1}{k}\binom{m-n_1}{n_2-k}}{\binom{m}{n_1}\binom{m}{n_2}} = \frac{\binom{n_1}{k}\binom{m-n_1}{n_2-k}}{\binom{m}{n_2}}.$$

Relationship between k pairs of links overlapped and a connection request blocked:

A connection request not blocked iff

$$n_1 + n_2 - k < m$$

which implies

$$k \geq \max\{0, n_1 + n_2 - m + 1\}$$

We also have

$$k \leq \min\{n_1, n_2\}$$

$$\Pr\{\textbf{connection not blocked} \mid \boldsymbol{n_1, n_2}\} = \frac{1}{\binom{m}{n_2}} \sum_{k=\max\{0,n_1+n_2-m+1\}}^{\min\{n_1,n_2\}} \binom{n_1}{k}\binom{m-n_1}{n_2-k}$$

Under the assumption that the events that individual links in the network are busy are independent

$$\Pr\{\boldsymbol{n_1}, \boldsymbol{n_2}\} = \Pr\{\boldsymbol{n_1}\} \cdot \Pr\{\boldsymbol{n_2}\}$$

We have

$$\Pr\{\boldsymbol{n_1}\} = \frac{\binom{m}{n_1} p^{n_1} q^{m-n_1}}{\sum_{j=0}^{n-1} \binom{m}{j} p^j q^{m-j}}$$

$$\Pr\{\boldsymbol{n_2}\} = \frac{\binom{m}{n_2} p^{n_2} q^{m-n_2}}{\sum_{j=0}^{n-1} \binom{m}{j} p^j q^{m-j}}$$

Blocking Probability

$$\Pr\{\text{connection not blocked}\}$$

$$= \frac{\sum_{n_1=0}^{n-1}\sum_{n_2=0}^{n-1}\frac{1}{\binom{m}{n_2}}\sum_{k=\max\{0,n_1+n_2-m+1\}}^{\min\{n_1,n_2\}}\binom{n_1}{k}\binom{m-n_1}{n_2-k}\binom{m}{n_1}p^{n_1}q^{m-n_1}\binom{m}{n_2}p^{n_2}q^{m-n_2}}{\left[\sum_{j=0}^{n-1}\binom{m}{j}p^j q^{m-j}\right]^2}$$

$$= \frac{\sum_{n_1=0}^{n-1}\sum_{n_2=0}^{n-1}\sum_{k=\max\{0,n_1+n_2-m+1\}}^{\min\{n_1,n_2\}}\binom{m}{n_1}\binom{n_1}{k}\binom{m-n_1}{n_2-k}p^{n_1+n_2}q^{2m-n_1-n_2}}{\left[\sum_{j=0}^{n-1}\binom{m}{j}p^j q^{m-j}\right]^2}$$

P_B = 1 - Pr{connection not blocked}

Theorem 1 ***The blocking probability of the Clos network*** P_B ***becomes zero when the number of middle stage switches*** $m \geq 2n - 1$**.**

Proof. The following equalities hold

$$\sum_{v=0}^{u} \binom{s}{v}\binom{t}{u-v} = \binom{s+t}{u}$$
$$\sum_{v=0}^{\min\{u,s\}} \binom{s}{v}\binom{t}{u-v} = \binom{s+t}{u} \qquad (1)$$

When $m \geq 2n-1$**, for any** n_1**,** n_2**,** $0 \leq n_1, n_2 \leq n-1$**,**

$$n_1 + n_2 \leq 2(n-1) \leq m-1,$$

which implies $\max\{0, n_1+n_2-m+1\} = 0$. **Using equality (1), we have**

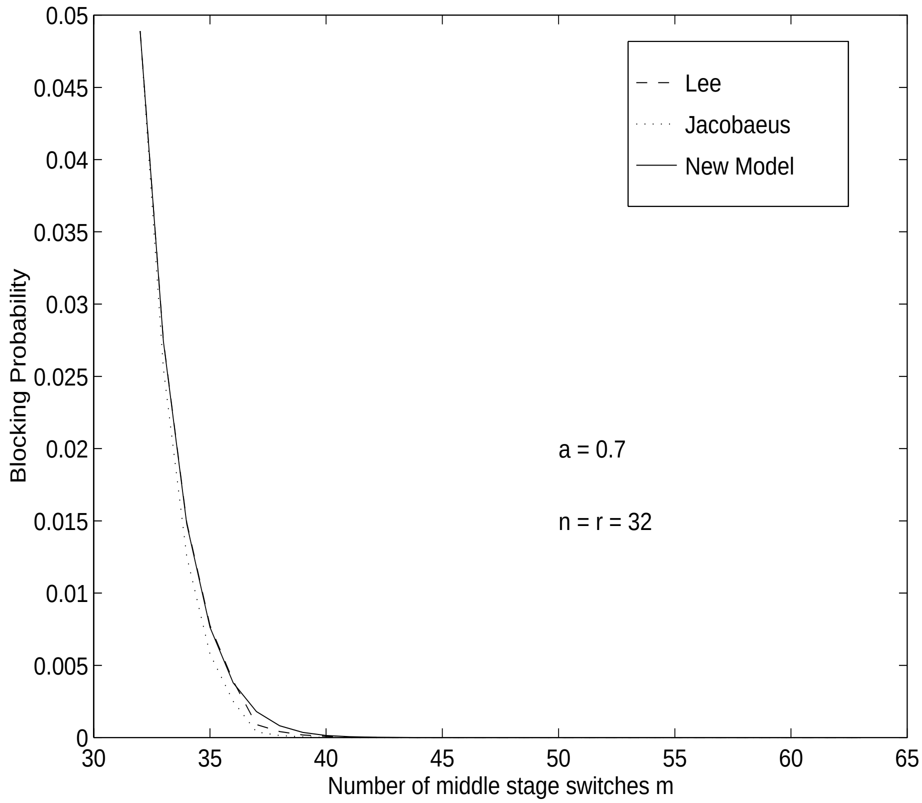
$$\frac{1}{\binom{m}{n_2}} \sum_{k=0}^{\min\{n_1,n_2\}} \binom{n_1}{k}\binom{m-n_1}{n_2-k} = 1.$$

$$\Pr\{\textbf{connection not blocked}\} = \frac{\sum_{n_1=0}^{n-1}\sum_{n_2=0}^{n-1} 1 \cdot \binom{m}{n_1} p^{n_1} q^{m-n_1} \cdot \binom{m}{n_2} p^{n_2} q^{m-n_2}}{\left[\sum_{j=0}^{n-1} \binom{m}{j} p^j q^{m-j}\right]^2} = 1$$

and $P_B = 1 - 1 = 0$.

Blocking Probability Comparison

$(n = r = 32)$



The blocking probabilities of the Clos network in three models: Lee, Jacobaeus and the new model, for a network with $n = r = 32$ and $n \leq m \leq 2n - 1$, under network input link utilization $a = 0.7$.

Blocking probability comparison, $a = 0.7$.

n	m	P_B(Lee)	P_B(Jacobaeus)	P_B(This paper)
64	64	0.002	0.002	0.002
64	68	0.0002	0.00016	0.0002
64	72	1.5×10^{-5}	6.2×10^{-6}	1.5×10^{-5}
64	76	8×10^{-7}	1.5×10^{-7}	8×10^{-7}
120	128	10^{-7}	3.5×10^{-8}	10^{-7}
240	256	10^{-14}	1.3×10^{-15}	1.3×10^{-14}

Experimental Simulations

- **Simulations were carried out for random routing strategy.**
- **Three network configurations were simulated:**

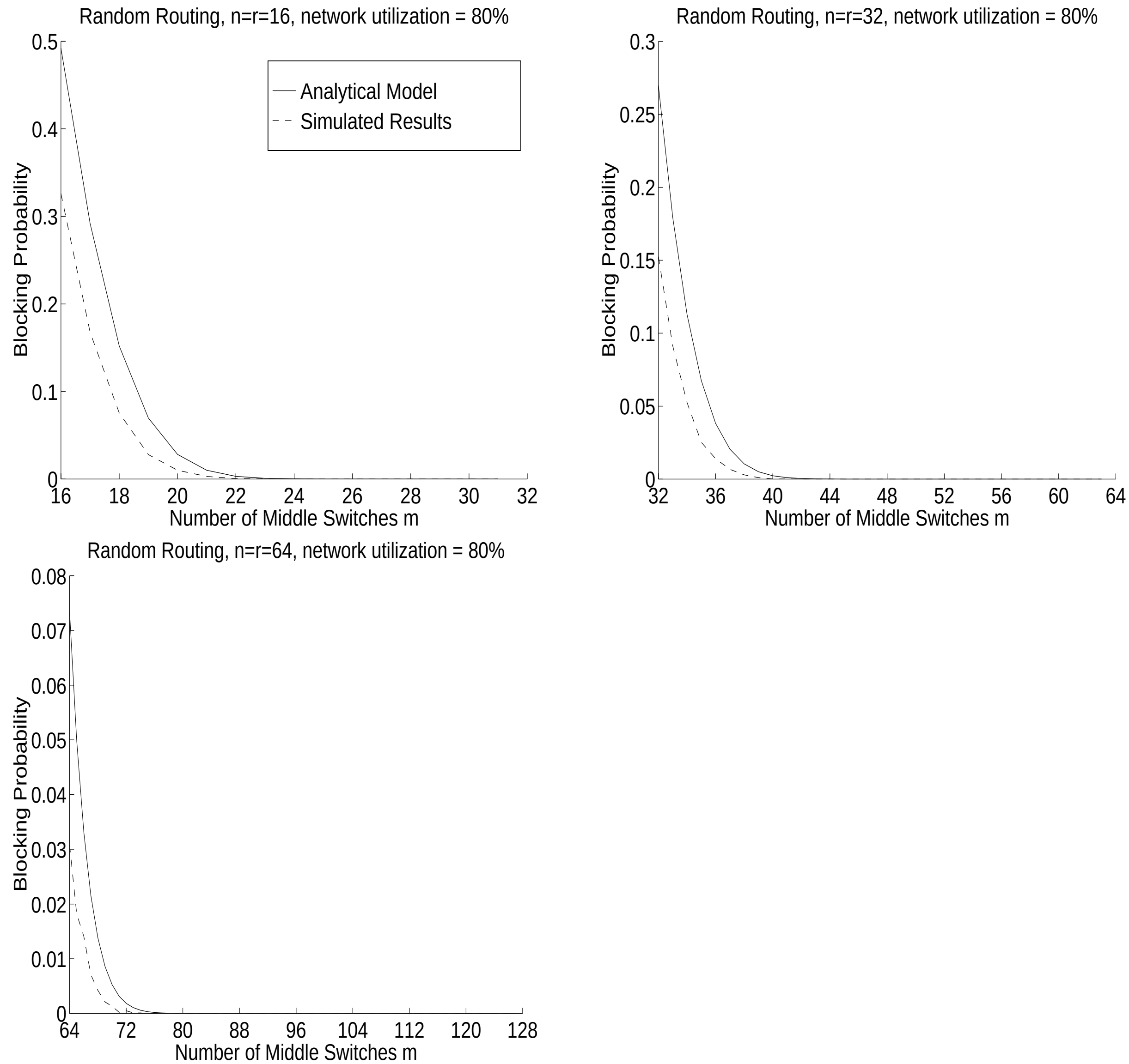
  $n = r = 16$, $16 \leq m \leq 31$

  $n = r = 32$, $32 \leq m \leq 63$

  $n = r = 64$, $64 \leq m \leq 127$

- **Network utilization:** $50\% \leq a \leq 90\%$
- **Fully packed switches:** $1 \leq d \leq 4$
- $10,000$ **connection requests processed per configuration.**

Analytical model vs. simulated results



$n = r = 16$ **with** $16 \leq m \leq 31$**,** $n = r = 32$ **with** $32 \leq m \leq 63$ **and** $n = r = 64$ **with** $64 \leq m \leq 127$ **at 80% network utilization.**

Summary:

Proposed a new analytical model on the blocking probability of the Clos networks under random routing strategy.

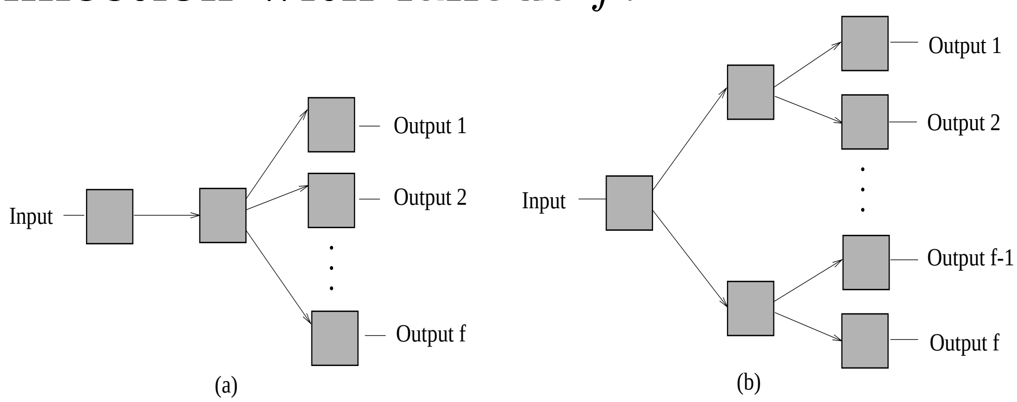
- **The newly proposed model can more accurately describe the blocking behavior of the network and is consistent with the well-known deterministic nonblocking condition.**
- **The analytical model is consistent with the blocking probabilities acquired through simulation.**
- **The new model may be extended to other routing strategies.**

Analytical model for the blocking probability of multicast Clos networks

- **The necessary and sufficient nonblocking condition obtained suggests that there is little room for further improvement on the multicast nonblocking condition.**
- **What is the blocking behavior of the multicast network with smaller number of middle stage switches? For example, a network with only the same number of middle stage switches as a nonblocking permutation network, i.e.** $m = 2n - 1$**.**
- **Develop an analytical model for the blocking probability of** $v(m, n, r)$ **multicast network.**
- **Look into the blocking behavior of the networks under various routing strategies through simulations to validate the model.**

The limitation of Lee's model when applied to multicast communication

- **Different ways to realize a multicast connection with fanout f.**



- **The total number of ways to realize a multicast connection with fanout f ($1 \le f \le r$) is**

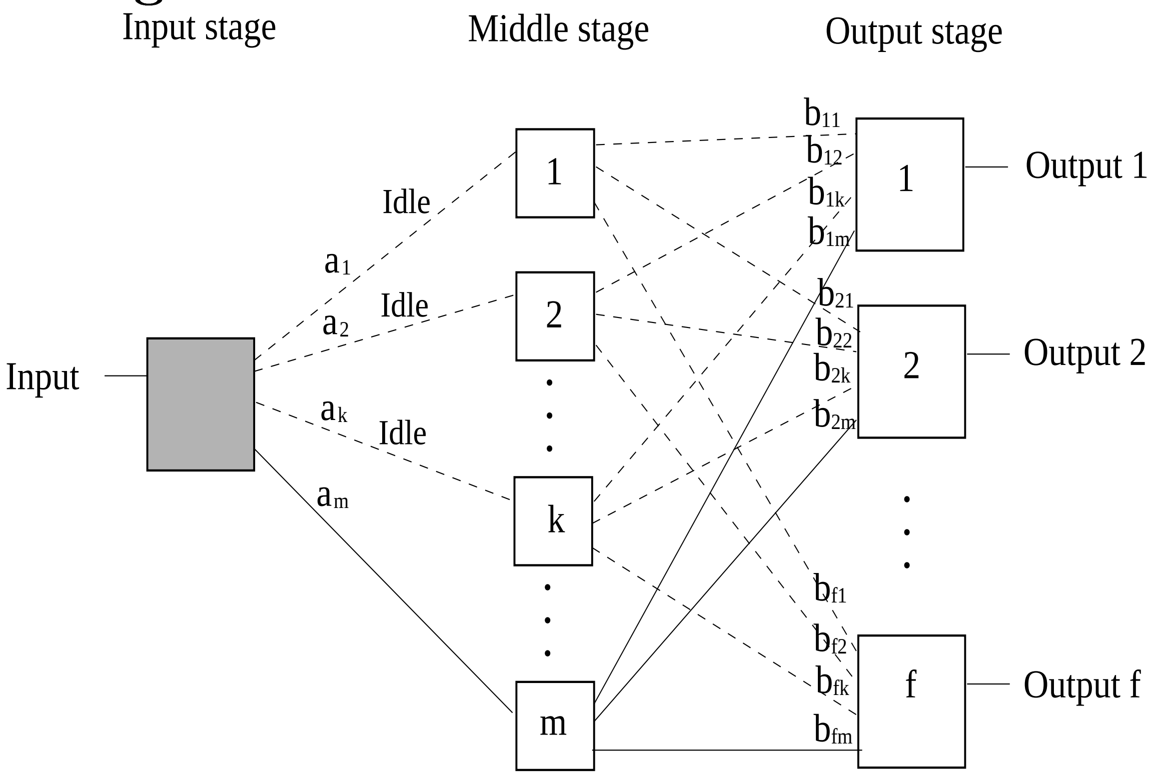
$$\sum_{j=1}^{f} \binom{m}{j} S(f, j) j!,$$

where $S(f, j)$ is the Stirling number of the second kind.

- **The dependencies among multicast trees make the problem intractable.**

Analytical model for multicast communication

A subnetwork associated with a multicast connection with fanout f, where k input-middle interstage links are idle.



Notations and assumptions

- **a$_i$: the event that the input-middle interstage link a_i is busy.**
- **b$_{ij}$: the event that the middle-output interstage link b_{ij} is busy.**
- **ε: the event that the connection request with fanout f cannot be realized.**
- **σ: the state of the input-middle interstage links $a_1, a_2, \dots, a_m$.**
- **$P(\varepsilon|\sigma)$: the conditional blocking probability in this state.**
- **$P(\sigma)$: the probability of being in state σ.**

$$P(\sigma) = q^k p^{m-k}$$

- **Still follow Lee's assumption that the events that individual links are busy are independent.**

Blocking probability for a multicast connection with fanout f

$$P_B(f) = P(\boldsymbol{\varepsilon}) = \sum_{\sigma} P(\sigma)P(\boldsymbol{\varepsilon}|\sigma)$$

$$= \sum_{k=0}^{m} \binom{m}{k} q^k p^{m-k} P(\boldsymbol{\varepsilon}|\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_m)$$

Blocking property of the subnetwork

Lemma 2 ***Assume that the interstage links*** $a_1, a_2, \ldots, a_k$ ***in the subnetwork are idle. A multicast connection from an input of the input switch to*** f ***distinct output switches cannot be realized if and only if there exists an output switch whose first*** k ***inputs are busy.***

- **Let ε' be the event that the connection request with fanout f cannot be realized given links $a_1, a_2, \ldots, a_k$ are idle.**

$$P(\boldsymbol{\varepsilon'}) = P(\boldsymbol{\varepsilon}|\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_m).$$

- **From Lemma 2, event ε' can be expressed in terms of events $\mathbf{b}_{ij}$'s:**

$$\begin{aligned} \boldsymbol{\varepsilon'} &= (\mathbf{b}_{11} \cap \mathbf{b}_{12} \cap \cdots \cap \mathbf{b}_{1k}) \\ &\quad \cup (\mathbf{b}_{21} \cap \mathbf{b}_{22} \cap \cdots \cap \mathbf{b}_{2k}) \cup \cdots \\ &\quad \cup (\mathbf{b}_{f1} \cap \mathbf{b}_{f2} \cap \cdots \cap \mathbf{b}_{fk}). \end{aligned}$$

- **The probability of event ε'**

$$\begin{aligned} P(\boldsymbol{\varepsilon'}) &= 1 - \prod_{i=1}^{f} P(\overline{\mathbf{b}_{i1} \cap \mathbf{b}_{i2} \cap \cdots \cap \mathbf{b}_{ik}}) \\ &= 1 - \prod_{i=1}^{f} [1 - P(\mathbf{b}_{i1} \cap \mathbf{b}_{i2} \cap \cdots \cap \mathbf{b}_{ik})] \\ &= 1 - \prod_{i=1}^{f} (1 - p^k) = 1 - (1 - p^k)^f \end{aligned}$$

Blocking probability for a multicast connection with fanout f

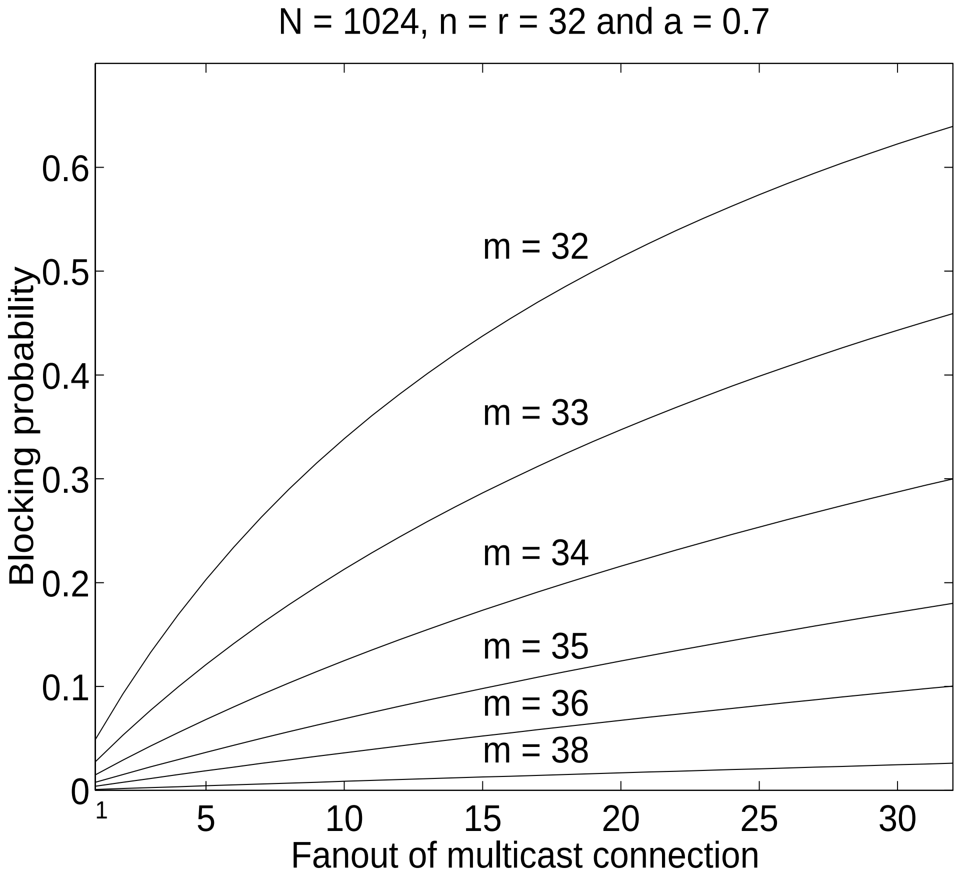
$$P_B(f) = \sum_{k=0}^{m} \binom{m}{k} q^k p^{m-k}[1-(1-p^k)^f].$$

Unicast special case ($f = 1$):

$$\begin{aligned} P_B(1) &= \sum_{k=0}^{m} \binom{m}{k} q^k p^{m-k}[1-(1-p^k)] \\ &= p^m \sum_{k=0}^{m} \binom{m}{k} q^k \\ &= p^m (1+q)^m \\ &= (1-q)^m (1+q)^m = (1-q^2)^m. \end{aligned}$$

This is exactly Lee's blocking probability for the $v(m, n, r)$ permutation network.

Blocking probabilities for $v(m, 32, 32)$ network with fanouts between 1 and 32:



The blocking probability $P_B(f)$ is an increasing sequence of fanout f.

Average blocking probability over all fanouts

- **Suppose the probability distribution for different fanouts in a multicast connection is**

$$\{w_f | 0 \leq w_f \leq 1, 1 \leq f \leq r, \sum_{i=1}^{r} w_i = 1\}.$$

- **The average value of the blocking probability, simply referred to as the** *blocking probability of the $v(m,n,r)$ multicast network***:**

$$P_B = \sum_{f=1}^{r} P_B(f) \cdot w_f.$$

- **Suppose the fanout is uniformly distributed over 1 to r.**

$$\begin{aligned} P_B &= \frac{1}{r} \sum_{f=1}^{r} P_B(f) \\ &= \frac{1}{r} \sum_{f=1}^{r} \sum_{k=0}^{m} \binom{m}{k} q^k p^{m-k} [1 - (1 - p^k)^f]. \end{aligned}$$

Asymptotic bound on the blocking probability

- **The following inequality holds**

$$1-(1-x)^l < lx,$$

where $0 < x < 1$, and l is an integer ≥ 1.

- **By applying the above inequality, we can obtain an upper bound on P_B:**

$$\begin{aligned} P_B &< \frac{1}{r}\sum_{f=1}^{r}\sum_{k=0}^{m}\binom{m}{k}q^k p^{m-k}\cdot f\cdot p^k \\ &= \frac{1}{r}(1-q^2)^m\sum_{f=1}^{r} f \\ &= \frac{r+1}{2}[1-(1-p)^2]^m \end{aligned}$$

Consider two cases:

Case 1: $m = n + c$**, for some constant** $c > 1$**.**

Note that $p = \frac{an}{m}$**, where** a **is a constant and** $0 \leq a < 1$**. Then**

$$[1-(1-p)^2]^m = \left[1-\left(1-\frac{an}{m}\right)^2\right]^m < [1-(1-a)^2]^m$$

which implies

$$P_B = O(r \cdot \delta^m),$$

where $\delta = 1-(1-a)^2$**.**

Case 2: $m = dn$**, for some constant** $d > 1$**.**

Since $p < \frac{n}{m} = \frac{1}{d}$**,**

$$P_B < \frac{r+1}{2}\left[1 - \left(1 - \frac{1}{d}\right)^2\right]^m = O(r \cdot \delta'^m),$$

where $\delta' = 1 - (1 - \frac{1}{d})^2$**. That is,** δ' **is a constant such that** $0 < \delta' < 1$**.**

In both cases, if $r = O(n)$ **we obtain**

$$P_B = O(e^{-\epsilon n})$$

where ϵ **is a constant** > 0**, which means the blocking probability tends to zero very quickly as** n **increases.**

More accurate blocking probability

- **In the case of $k > n-1$, there must exist some idle input on each of f output switches.**
- **The condition blocking probability**

$$
\begin{aligned}
& P(\boldsymbol{\varepsilon}|\bar{\mathbf{a_1}}, \ldots, \bar{\mathbf{a_k}}, \mathbf{a_{k+1}}, \ldots, \mathbf{a_m}) \\
= & \begin{cases} 1-(1-p^k)^f & \textbf{if } 1 \le k \le n-1 \\ 0 & \textbf{if } k \ge n. \end{cases}
\end{aligned}
$$

- **The blocking probability for a multicast connection with fanout f**

$$
\begin{aligned}
P_B(f) &= \sum_{k=m-n}^{m} \binom{m}{k} q^k p^{m-k} [1-(1-p^k)^f] \\
&= \sum_{i=0}^{n} \binom{m}{m-n+i} q^{m-n+i} p^{n-i} [1-(1-p^{m-n+i})^f].
\end{aligned}
$$

Comparison between the old $P_B(f)$ and the new $P_B(f)$ for $n = 32, m = 64, r = 32$, and $a = 0.7$.

Fanout f	$P_B(f)$ **(old)**	$P_B(f)$ **(new)**
1	5.46×10^{-16}	2.77×10^{-17}
2	1.09×10^{-15}	5.55×10^{-17}
5	2.74×10^{-15}	1.46×10^{-16}
8	4.38×10^{-15}	2.33×10^{-16}
12	6.57×10^{-15}	3.51×10^{-16}
16	8.76×10^{-15}	4.66×10^{-16}
20	1.10×10^{-14}	5.87×10^{-16}
24	1.31×10^{-14}	6.99×10^{-16}
28	1.53×10^{-14}	8.12×10^{-16}
32	1.75×10^{-14}	9.33×10^{-16}

Generalization to asymmetric Clos networks

- **Asymmetric Clos type network or**
  $v(m, n_1, r_1, n_2, r_2)$ **network**
  n_1: **number of inputs on each input switch**
  r_1: **number of input switches**
  n_2: **number of outputs on each output switch**
  r_2: **number of output switches**
- $P(\bar{\mathbf{a}}_{\mathbf{i}}) = p_a = \frac{an_1}{m}$, $q_a = 1 - p_a$
- $P(\bar{\mathbf{b}}_{\mathbf{i,j}}) = p_b = \frac{an_2}{m}$, $q_b = 1 - p_b$
- **Blocking probability for a multicast connection with fanout** f

$$P_B(f) = \sum_{k=0}^{m} \binom{m}{k} q_a^k p_a^{m-k} [1 - (1 - p_b^k)^f]$$

Experimental study

Extensive simulations were carried out for seven routing control strategies.

Definitions:

- **Connection request I_i:**
  the output switches to be connected from input port i in a multicast connection.

- **Available middle switches of input port i:**
  the set of middle switches with idle links to input port i.

- **Destination set of a middle switch:**
  busy outputs of a middle switch.

A generic routing algorithm

Step 1: If no available middle switches for the current connection request, then exit.

Step 2: Choose a non-full middle switch among the available middle switches for the connection request according to some control strategy. If no such middle switch exit.

Step 3: Realize as large as possible portion of the connection request in the middle switch chosen in Step 2.

Step 4: Update the connection request by discarding the portion that is satisfied by the middle switch chosen in Step 2.

Step 5: If the connection request is non-empty, go to Step 1.

Routing control strategies

1. **Smallest Absolute Cardinality Strategy**
2. **Largest Absolute Cardinality Strategy**
3. **Average Absolute Cardinality Strategy**
4. **Smallest Relative Cardinality Strategy**
5. **Largest Relative Cardinality Strategy**
6. **Average Relative Cardinality Strategy**
7. **Random Strategy**

Model assumptions

- **Three types of traffic distributions are considered: uniform traffic, uniform/constant, and Poisson traffic.**
- **In the steady state, the arrival rate of the connection requests is approximately equal to the departure rate (service rate) of the connections.**
- **A new multicast connection request is randomly generated among all idle network input ports and output ports.**

- **During the network operation, a certain workload is maintained. The workload is measured by the network utilization, which is defined as**

$$\textbf{Utilization} = \frac{\textbf{The total number of busy output ports}}{N}$$

- **The blocking probability in the simulation is computed by**

$$P_B = \frac{\text{The total number of connection requests blocked}}{\text{The total number of connection requests generated}}$$

Experimental simulations

- **Two network configurations:**

  $N = 1024, n = r = 32$**, and** $32 \leq m \leq 48$**.**

  $N = 4096, n = r = 64$**, and** $64 \leq m \leq 84$**.**

- **Seven routing control strategies**

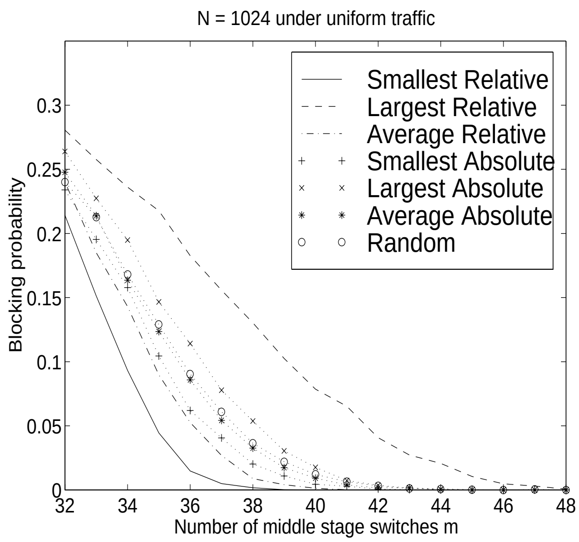
- **Three types of traffic: uniform, uniform/constant, and Poisson**

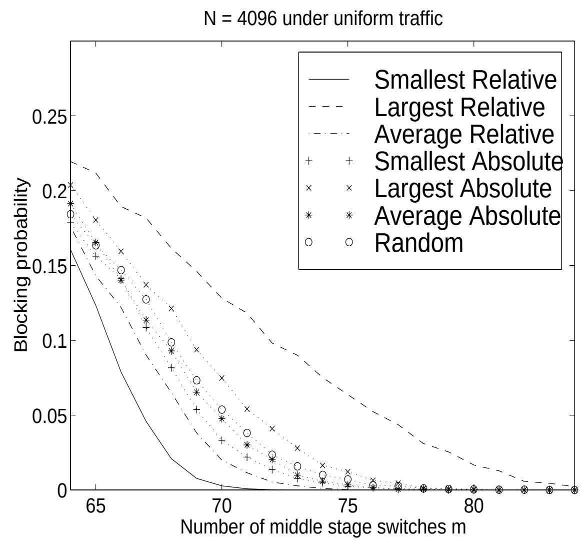
- **Initial network utilization** $= 90\%$

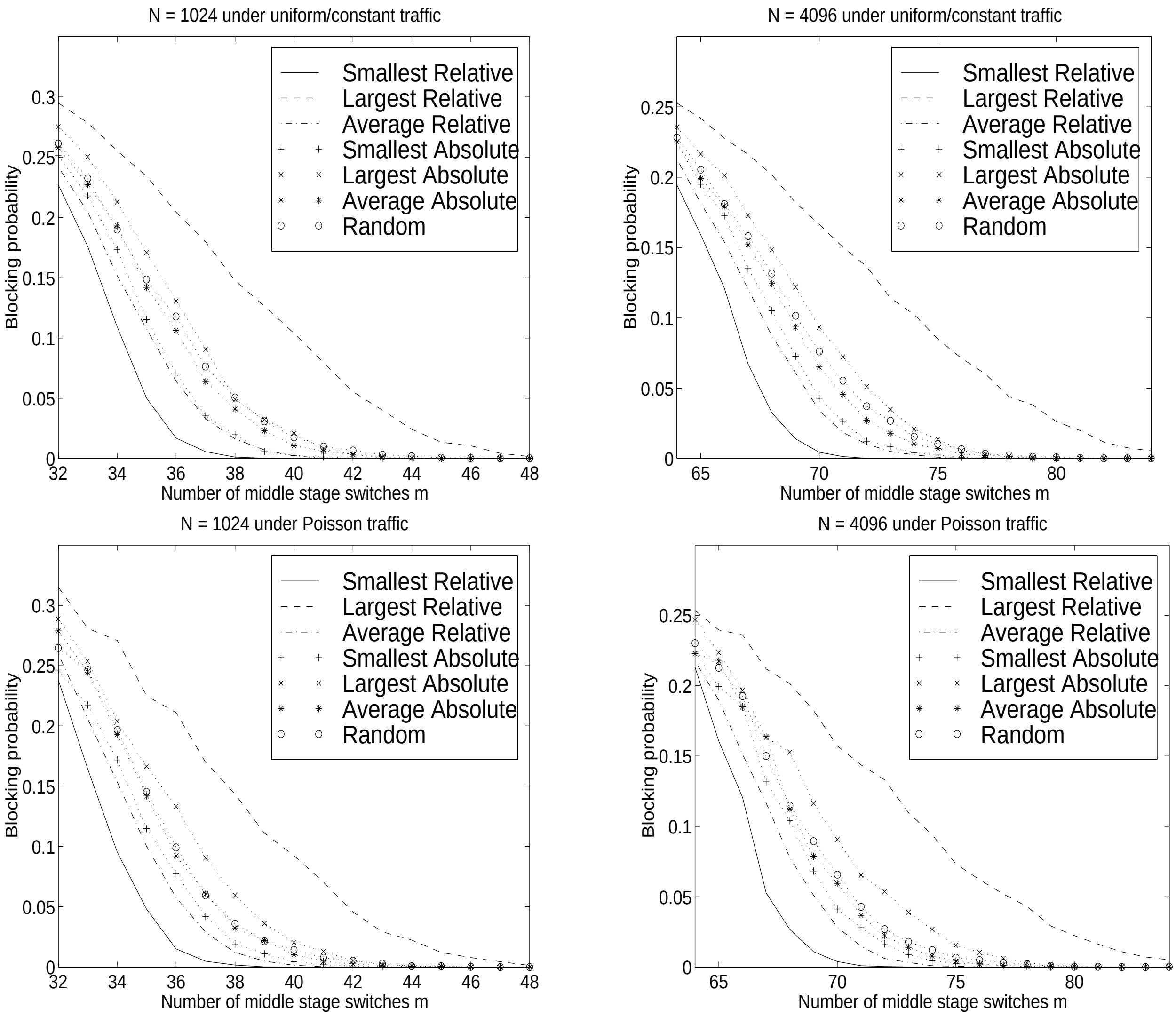
- $25,000$ **connection requests processed per configuration per strategy**

Simulation results

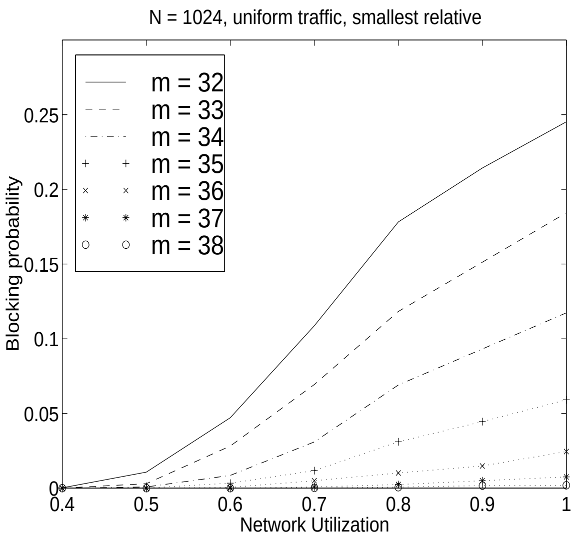
The blocking probability of the $v(m, n, r)$ multicast network under seven routing control strategies:

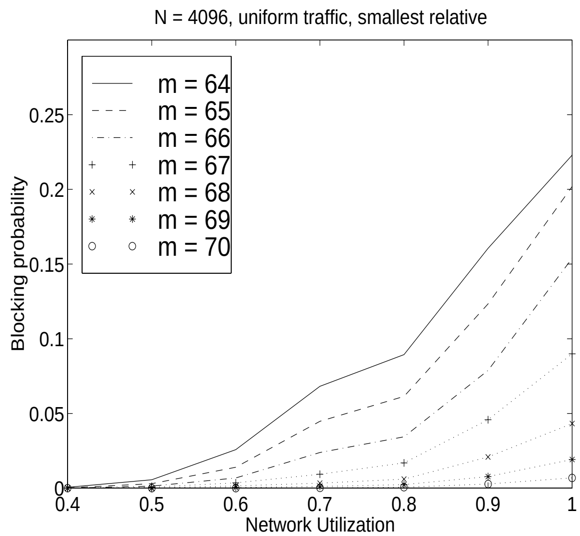


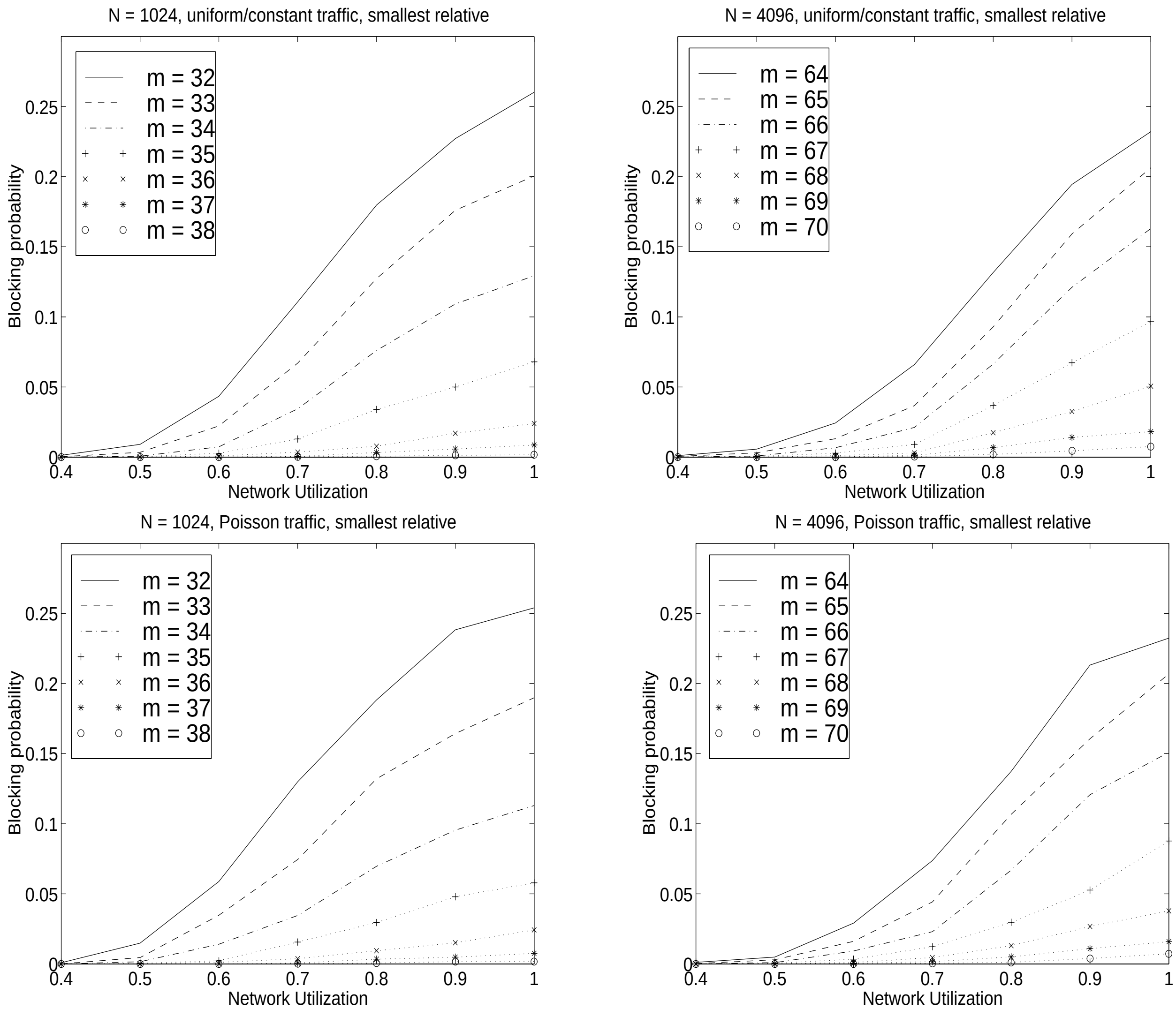



N = 1024 under uniform/constant traffic
N = 4096 under uniform/constant traffic
N = 1024 under Poisson traffic
N = 4096 under Poisson traffic
Smallest Relative
Largest Relative
Average Relative
Smallest Absolute
Largest Absolute
Average Absolute
Random
Blocking probability
Number of middle stage switches m

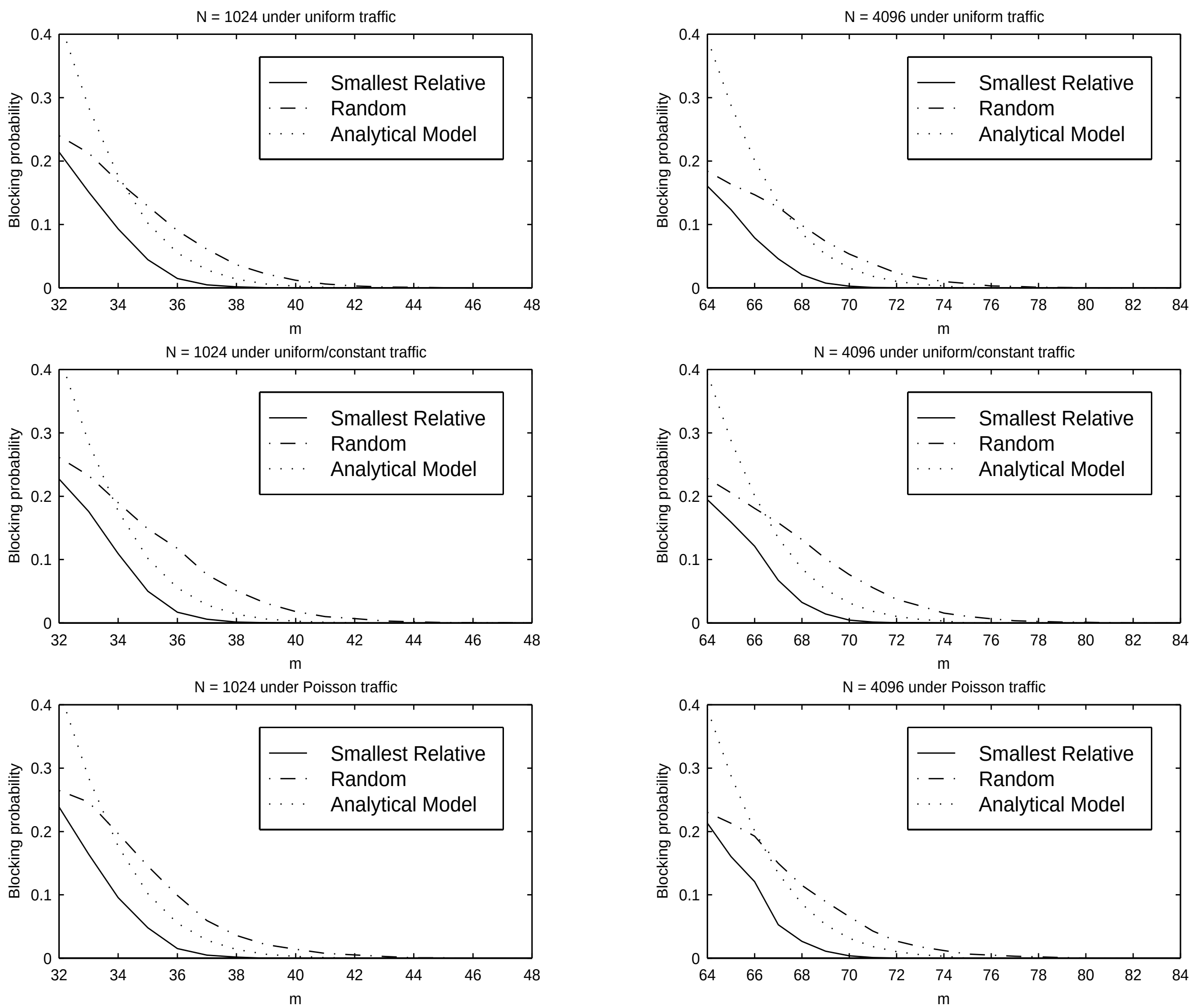
The blocking probability of the $v(m, n, r)$ multicast networks under different network utilization for the smallest relative strategy






N = 1024, uniform/constant traffic, smallest relative
N = 4096, uniform/constant traffic, smallest relative
N = 1024, Poisson traffic, smallest relative
N = 4096, Poisson traffic, smallest relative
Blocking probability
Network Utilization
m = 32
m = 33
m = 34
m = 35
m = 36
m = 37
m = 38
m = 64
m = 65
m = 66
m = 67
m = 68
m = 69
m = 70
0.25
0.2
0.15
0.1
0.05
0
0.4
0.5
0.6
0.7
0.8
0.9
1

Comparison between the analytical model and the simulation results



Summary:

Studied the blocking behavior of the multicast Clos network along two parallel lines:

- **developed an analytical model for the blocking probability of the multicast Clos network;**
- **studied the blocking behavior of the network under various routing control strategies through simulations.**

Observations:

- **A network with a small m, such as $m = n + c$ or dn, is almost nonblocking for multicast connections, although theoretically it requires $m \geq \Theta\left(n\frac{\log r}{\log\log r}\right)$ to achieve nonblocking for multicast connections.**

- **Routing control strategies are effective for reducing the blocking probability of the multicast network. The best routing control strategy can provide a factor of 2 to 3 performance improvement over random routing.**

- **The results indicate that a Clos network with a comparable cost to a permutation network can provide cost-effective support for multicast communication.**