DOA estimation with k-times extended co-prime arrays

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***Abstract*—Sparse arrays such as co-prime arrays, nested arrays and minimum redundancy arrays (MRAs) can achieve larger number of degrees-of-freedom (DoFs) with fewer sensors by exploring their difference co-arrays. Co-prime arrays become more attractive, as there exist no exact expressions of MRA configurations for an arbitrary number of sensors and the nested arrays suffer greatly from mutual coupling. However, in order to achieve a large number of DoFs, traditional extended co-prime arrays have a large array aperture compared to the DoFs gained. In this paper, we propose a novel advanced co-prime array geometry that can not only further reduce the number of required sensors but also significantly reduce the array aperture while achieving the same number of DoFs. On the other hand, it has better performance than nested arrays in the presence of mutual coupling. Simulation results demonstrate the feasibility and effectiveness of our proposed array geometry in achieving higher quality direction-of-arrival (DOA) estimation.**

I. Introduction

Array signal processing is one of the most important topics in signal processing. It can exploit the temporal and spatial data collected by the array of receiving sensors to estimate unknown parameters of the signal sources. Among these parameters, finding the direction-of-arrival (DOA) of sources is of critical importance, and can see its use in the areas of radar, sonar, astronomy, and wireless communications. Traditionally, a uniform linear array (ULA) with $N+1$ elements can identify N sources at most, and has a degree of freedom (DoF) of N. However, detecting a large number of sources using ULA requires a large number of sensors, which would incur a big cost.

To address this challenge, sparse arrays such as minimum redundancy arrays (MRAs) [1], co-prime arrays [2] and nested arrays [3] have been proposed. With fewer elements, these sparse-array techniques exploit difference co-arrays calculated from the correlation of the received signals to generate a longer consecutive virtual array for a larger number of DoFs. Generally, a sparse array with $O(N)$ sensors can offer $O(N^2)$ freedoms. However, sparse arrays have inevitable shortcomings as well. Without closed-form expressions of the array geometries and the number of achievable DoFs for different numbers of sensors, it is complicated to design an MRA system in practice. Although co-prime arrays have definite expressions, their difference co-arrays contain holes, which significantly reduce the number of achievable DoFs. Nested arrays do not have the hole problem, but they suffer more from mutual coupling because they have a dense ULA segment in their physical configurations.

Some recent efforts have been made to alleviate these issues. In Pal *et. al* [4], an extended co-prime array structure is introduced to lengthen the consecutive segment in the virtual array. With more sensors in the array, the efficiency of the array reduces about 50%. The work in [5] uses proportional frequencies to solve the hole problem in the virtual array. Besides the need of additional frequencies, these frequencies may not be available at the sources. Compressive sensing [6] is exploited to estimate the DOAs in the presence of holes, without the benefit of high DoFs enabled by a continuous virtual array. Authors in [7] propose to apply temporal signal coherence (TCP) in moving co-prime arrays to fill in the holes, while the precise temporal coherence is difficult to achieve in the practical environment. Qin *et. al* [8] introduce a generalized co-prime array concept. It puts constraints on the selection of M and N, and also can not generalize all co-prime array geometries. Liu *et. al* propose super nested arrays [9] to reduce the mutual coupling effect in nested arrays. Without enough redundancy in the virtual arrays, the performance of super nested arrays may deteriorate significantly in the presence of sensor localization inaccuracies.

In this work, we propose a new array infrastructure, named the *k-times extended co-prime array*. It can form a longer ULA segment in the virtual array while not compromising the array efficiency. Compared with traditional co-prime arrays, it can achieve a larger number of DoFs when having the same number of sensors. Alternatively, to obtain the same number of DoFs, it requires much fewer physical sensors and also a smaller array aperture. The significant reduction of the array aperture will make it better fit for use in a space-constrained scenario such as air-borne applications. On the other hand, compared with the hole-free nested array, it has better performance in the presence of mutual coupling between array elements.

The remainder of the paper is organized as follows. In section 2, we review the configurations of traditional co-prime arrays. Section 3 introduces our proposed k-times extended co-prime array geometry and studies its theoretical properties. In section 4, we present our signal models and the DOA estimation process. We provide performance studies with simulation results in Section 5 and conclude the work in Section 6.

II. Review of Co-prime Arrays

A conventional co-prime array [2] shown in Fig. 1 consists of two uniform linear sub-arrays with the separation Md and Nd respectively. There are N sensors in the first sub-array and M sensors in the second sub-array. M and N are co-prime integers, i.e., $gcd(M, N) = 1$, and d is the unit of inter-element spacing. To avoid spatial aliasing, d is typically set to $\lambda/2$, where λ is the wavelength of impinging narrowband signals. Since the first sensors of the two uniform linear sub-arrays are co-located, the total number of sensors in conventional co-prime array is $M + N - 1$.

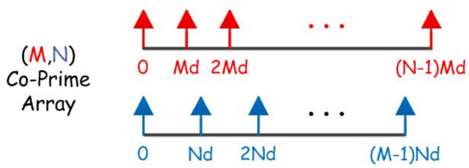


Fig. 1: Structure of Conventional Co-prime Array

In the sparse array signal processing, to achieve a given number of DoFs with fewer physical sensors, a virtual array generated from the difference co-array is applied to substitute for the physical array. However, elements in the difference co-array are often inconsecutive and leave holes. This reduces the length of the ULA segment in the virtual array and thus the effective number of DoFs. For example, the difference co-array of a $(3, 4)$ conventional co-prime array can be represented by the set $[-9d, +9d]$ except $\pm 7d$. As a result, an (M, N) conventional co-prime array usually can not obtain MN DoFs.

III. k-times Co-prime Arrays Geometry

A. Array Structure

To effectively increase the number of DoFs, we propose a new co-prime array geometry, called *k-times extended co-prime array* to alleviate the hole problem. The array structure of (M, N) k-times extended co-prime array is shown in Fig. 2. Different from the conventional co-prime array, it has k times the number of sensors in the second uniform linear sub-array.

The locations of sensors in the (M, N) k-times extended co-prime array can be described by the set:

$$\mathbb{L}_C = \{Mnd\} \cup \{Nmd\}, \tag{1}$$

where $0 \le n \le N - 1$ and $0 \le m \le kM - 1$.

The total number of sensors is $kM + N - 1$ and the array aperture is $(kM - 1)Nd$.

B. Challenges and Properties

In order to determine the capability of an (M, N) k-times extended co-prime array, we need to answer a set of questions:

1) Can this new array structure increase the number of DoFs given that it requires a larger number of sensors and has a larger array aperture?
2) How to determine the number of DoFs? Is there a close-form equation to represent DoFs for a given set of parameters M, N, k?
3) Does the increase of DoFs weigh over the increase of the number of sensors and the aperture? How to evaluate the efficiency of the array?
4) How does the selection of (M, N) and k affect the performance?

Intuitively, adding more sensors in the second sub-array would help generate a longer consecutive ULA segment in the virtual array. Before answering the above set of questions, we first give a lemma below.

Lemma 1. *Given an integer p in the range of $0 \le p \le (k - 1)MN + M - 1$ and $gcd(M, N) = 1$, we can always find two integers n_1 and n_2 in the range of $0 \le n_1 \le N - 1$ and $0 \le n_2 \le kM - 1$, such that $p = Nn_2 - Mn_1$.*

Proof. From the Euclidean theorem, for an integer p, we can always find two integers n'_1 and n'_2 such that

$$p = Nn'_2 - Mn'_1 \tag{2}$$

when $gcd(M, N) = 1$. We can rewrite p as

$$p = N(n'_2 - qM) - M(n'_1 - qN) \tag{3}$$

for any integer q. We then define $n_1 \triangleq n'_1 - qN$ and $n_2 \triangleq n'_2 - qM$. We can choose an appropriate q such that n_1 is in the range $0 \le n_1 \le N - 1$. From equation(3), we can get

$$Nn_2 = p + Mn_1 \tag{4}$$

Now we substitute $0 \le n_1 \le N - 1$ and $0 \le p \le (k - 1)MN + M - 1$ into equation(4), we have

$$0 \le Nn_2 \le kMN - 1 < kMN \tag{5}$$

and therefore we can get $0 \le n_2 < kM$ that is $0 \le n_2 \le kM - 1$. □

Lemma 1 demonstrates that a k-times extended co-prime array has a ULA segment without holes in the range of $[0, (k - 1)MN + M - 1]$. From the symmetry of the difference co-array, the total length of the consecutive ULA segment is $2(k - 1)MN + 2M - 1$. After the spatial smoothing process in the estimation process to be introduced, only half of the virtual array can be approached. Therefore, the number of DoFs is $(k-1)MN+M-1$. Compared with the DoF number $MN+M-1$ in the traditional extended co-prime array, the number of DoFs indeed increases by about $k - 1$ times.

To evaluate the efficiency of different co-prime array structures, we count the ratio of the ULA segment in the difference co-array (i.e, the virtual array aperture):

$$\mathrm{r} = \frac{2(k - 1)MN + 2M - 1}{2kMN - 2N + 1}. \tag{6}$$

For a fixed k, we have

$$\lim_{M,N \to +\infty} \mathrm{r} = \frac{k - 1}{k} \tag{7}$$

This indicates that, with the increase of k, the ratio increases and a higher spatial efficiency can be achieved.

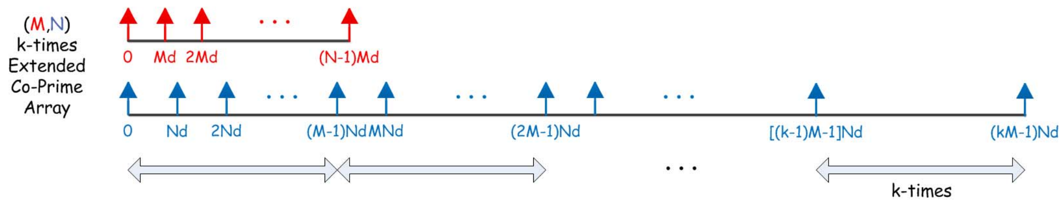


Fig. 2: Structure of k-times Extended Co-prime Array

TABLE I: Numerical Examples

$\lvert T\rvert$	(M,N)	k	DoF	Aperture(A)	DoF/A
21	$(7,8)$	2	62	104	59.62%
21	$(5,7)$	3	74	98	75.51%
20	$(2,7)$	7	85	91	93.40%
21	$(1,11)$	11	110	110	100%

C. Numerical Study

TABLE I shows several numerical examples under the constraint that the number of physical sensors is not greater than 21. When $k = 2$, the k-times extended co-prime array degenerates into the traditional extended co-prime array and has the smallest number of DoFs. As k increases, both the number of DoFs and the array efficiency increase. When $M = 1$ and $N = k$, the k-times extended co-prime array evolves into the nested array with the largest number of DoFs.

IV. Signal Models and DOA Estimation

In this section, we would like to show how DOAs can be estimated with our proposed k-times extended co-prime arrays. Assuming D narrowband sources with powers $[\sigma_1^2\ \sigma_2^2\ \cdots\ \sigma_D^2]$ impinge on the array from directions $[\theta_1\ \ \theta_2\ \ \cdots\ \ \theta_D]$, the signals received at the array elements can be expressed as

$$\mathbf{x}[t] = \mathbf{A}\mathbf{s}[t] + \mathbf{n}[t] \tag{8}$$

where $\mathbf{A}$ is the array manifold matrix of the form

$$\mathbf{A} = [\mathbf{a}(\theta_1)\ \mathbf{a}(\theta_2)\ \cdots\ \mathbf{a}(\theta_D)] \tag{9}$$

$$\text{with} \quad \mathbf{a}(\theta_i) = [e^{j\frac{2\pi d}{\lambda}l_1 sin\theta_i}, \cdots, e^{j\frac{2\pi d}{\lambda}l_{kM+N-1} sin\theta_i}]^T \tag{10}$$

where $[l_1, l_2, \cdots, l_{kM+N-1}]$ represent the locations of $kM + N - 1$ physical sensors. $\mathbf{s}[t] = [\mathbf{s}_1(t)\ \mathbf{s}_2(t)\ \cdots\ \mathbf{s}_D(t)]^T$ denotes the t^{th} snapshot of the source signal vector, and the noise vector $\mathbf{n}[t]$ is assumed to be temporally and spatially white and uncorrelated from the source.

In array signal processing, the difference co-array is formed naturally by finding the correlation of the received signal,

$$\mathbf{R}_{xx} = E[\mathbf{x}(t)\mathbf{x}(t)^H] = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma_n^2\mathbf{I}, \tag{11}$$

where $\mathbf{R}_{ss}$ is the source autocorrelation matrix, with

$$\mathbf{R}_{ss} = diag([\sigma_1^2\ \sigma_2^2\ \cdots\ \sigma_D^2]. \tag{12}$$

In practice, the autocorrelation matrix can be computed by the following sample average

$$\widehat{\mathbf{R}}_{xx} = \frac{1}{L}\sum_{t=1}^{L}\mathbf{x}(t)\mathbf{x}(t)^H, \tag{13}$$

where L is the total number of snapshots. In order to build the new model using the difference co-array as the new array manifold matrix, we vectorize the autocorrelation matrix

$$\mathbf{z} = vec(\mathbf{R}_{xx}) = \mathbf{B}\cdot\mathbf{p} + \sigma_n^2 vec(\mathbf{I}), \tag{14}$$

where $\mathbf{B} = [\mathbf{B}_{\theta_1}\ \mathbf{B}_{\theta_2}\ \cdots\mathbf{B}_{\theta_D}] = \mathbf{A}^* \odot \mathbf{A}$ (Khatri-Rao product of $\mathbf{A}^*$ and $\mathbf{A}$) and $\mathbf{p} = [\sigma_1^2\ \sigma_2^2\ \cdots\ \sigma_D^2]^T$.

We consider the vector $\mathbf{z}$ to be the new received data, $\mathbf{B}$ to be the new array manifold matrix and $\mathbf{p}$ to be the new source signal.

As there exist redundant and out-of-order elements in the vector, we have to drop and reorder some elements to rebuild $\mathbf{z}$ to form a new vector $\mathbf{z}'$ so that its corresponding $\mathbf{B}'$ has the same expression as the manifold of the ULA segment in the virtual array. The rebuilt vector $\mathbf{z}'$ can be expressed as

$$\mathbf{z}' = \mathbf{B}'\cdot\mathbf{p} + \mathbf{n}'. \tag{15}$$

Since the new source signals $\mathbf{p}$ are no longer incoherent, we use spatial smoothing technique [10] to build the rank of a positive semi-definite matrix from this new model. We divide the new received data vector $\mathbf{z}'$ into multiple vectors $\mathbf{z}_i^{'}$ so that its corresponding virtual ULA array is divided into multiple overlapping sub-arrays. Then we compute the autocorrelation-like matrix of each divided received data vector $\mathbf{z}_i^{'}$

$$\mathbf{R}_{zi} \triangleq \mathbf{z}_i'\mathbf{z}_i'^H \tag{16}$$

Taking the average of the autocorrelation matrices of all sub-arrays, we can get the final spatial smoothed matrix

$$\mathbf{R}_{zz} = \frac{1}{DoF}\sum_{i=1}^{DoF}\mathbf{R}_{zi}, \tag{17}$$

where DoF equals the number of sub-arrays and denotes the maximum number of detectable sources.

Finally, we can accomplish DOA estimation by utlizing MUltiple SIgnal Clasification (MUSIC) algorithm [11] on $\mathbf{R}_{zz}$.

V. Mutual Coupling Model

The signal model in equation (8) assumes the coupling is free, that is, the sensors in the array will not interfere with each other. However in practical antenna arrays, electromagnetic interaction between the antenna elements always exists. The radiation pattern in each antenna element of an array depends on its own excitation and also the contributions from adjacent antenna elements. The effect of mutual coupling is inversely proportional to the spacing between the different antenna elements in an array. The signal model in the presence of mutual coupling can be expressed as:

$$\mathbf{x}[t] = \mathbf{C}\mathbf{A}\mathbf{s}[t] + \mathbf{n}[t] \tag{18}$$

where $\mathbf{C}$ denotes the mutual coupling matrix and $\mathbf{C} = \mathbf{I}$ is a coupling-free case.

To determine the mutual coupling matrix $\mathbf{C}$, Z-approach [12] can be applied. The mutual coupling matrix $\mathbf{C}$ under Z-approach can be written as:

$$\mathbf{C} = \mathbf{Z}^{-1} \tag{19}$$

where $\mathbf{Z}$ is the load impedance matrix between voltages and currents of antenna terminals.

Although increasing k can enhance the spatial efficiency of the virtual array, the mutual coupling effect will also intensify because the distances between some antenna elements become smaller. The performance of the k-times array is determined by the appropriate selection of (M, N, k) based on the practical application requirement. The results in the next section will show the tradeoffs.

VI. Simulation Results

We evaluate the performance of our proposed k-times extended co-prime (k-ECPA) through MATLAB simulations. We run MUSIC algorithm to detect the DOAs of a group of uniformly distributed sources. We compare the performance of k-ECPA with k and (M, N) varied and with that of two reference methods: the traditional extended co-prime array (ECPA) and the nested array respectively.

A. MUSIC Spectrum

We consider a k-ECPA configuration with $k = 4$ and $(M, N) = (2, 5)$. The total number of sensors is $kM + N - 1 = 12$ and the array aperture is $(kM - 1)Nd = 35d$. The length of the ULA segment in the virtual array is $2(k-1)MN + 2M - 1 = 63$ and the achievable number of DoFs after utilizing the spatial smoothing technique is 31. If 12 sensors are used in the traditional extended co-prime array, we have $(M, N) = (4, 5)$, and the number of DoFs is $MN + M - 1 = 23$. We generate 26 sinusoidal sources with the SNR set to 0 dB and angles θ_i uniformly distributed within the range $[-60°, 60°]$. The covariance matrix is estimated using 2000 snapshots. DOA estimation with traditional ECPA will fail since the number of sources exceeds the number of DoFs. However, from the MUSIC spectrum in Fig. 3, we can see that our proposed k-ECPA can still successfully identify all 28 sources, which demonstrates the feasibility of our proposed array geometry in identifying a larger number of sources with the same number of physical sensors.

B. Root-Mean-Sqared Error (RMSE)

We then compare the accuracy of DOA estimation among several different array geometries with the calculation of the root-mean-squared error (RMSE):

$$RMSE = \sqrt{\frac{1}{D}\sum_{i=1}^{D}(\widehat{\theta}_i - \theta_i)^2} \tag{20}$$

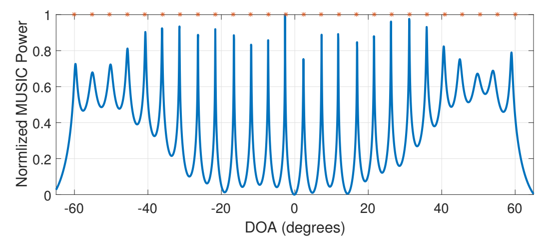
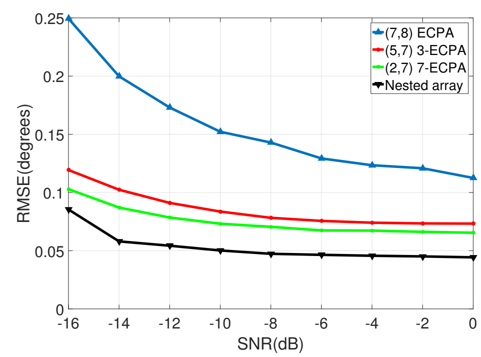
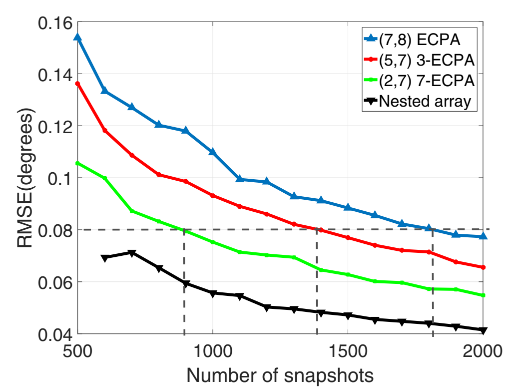


Fig. 3: MUSIC spectrum

through 100 Monte Carlo simulations, with the number of sensors set to $|T| \leq 21$ and the number of target sources set to $D = 45$.



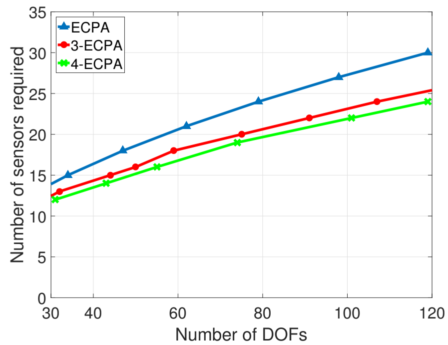
(a) RMSE vs SNR



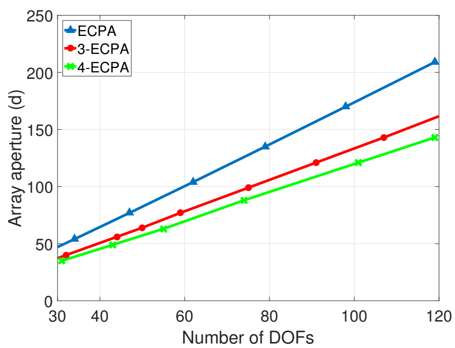
(b) RMSE vs Snapshots

Fig. 4: Root-Mean-Squared Error

Fig. 4 (a) shows RMSE as a function of SNR with 1000 snapshots. Compared with ECPA, our proposed k-ECPA can reduce the RMSE over 40%. Furthermore, it is clear that with the increase of k, RMSE is reduced and better detection performance is achieved. This is because a larger k allows for a larger number of DoFs. Fig. 4 (b) shows RMSE as a function of the number of snapshots with SNR set to 0dB. For a fixed RMSE threshold ($0.08°$), our proposed k-ECPA geometry with $k = 3$ and $k = 7$ can reduce the required number of snapshots by 25% and 50% respectively compared with ECPA.



(a) Number of sensors vs DoFs



(b) Array aperture vs DoFs

Fig. 5: Physical array parameters

C. Physical array parameters

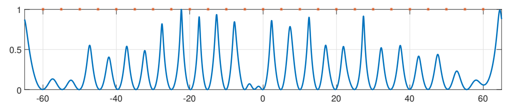
The physical array parameters among different geometries are compared in Fig. 5 after normalizing the number of DoFs. Compared with ECPA, 3-ECPA and 4-ECPA require up to 15% and 20% fewer physical sensors and reduce the array aperture up to 25% and 35%. This reduction is more remarkable as the number of DoFs increases.

D. MUSIC Spectrum under Mutual Coupling

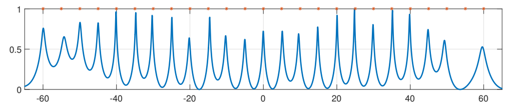
Finally, we take into consideration the mutual coupling resulted from the electromagnetic interaction between different antenna elements in the array. We compare $(3,4)$ 4-ECPA with nested array, where both methods have 15 sensors. 25 sources are uniformly distributed within the range $[-60°, 60°]$. SNR is set to 0dB and the number of snapshots is 500. The MUSIC spectrum under mutual coupling is shown in Fig. 6. Nested array fails to detect many sources with their angles mismatched, while our $(3,4)$ 4-ECPA can successfully detect most of the sources except the one at the angle $55°$. In the presence of mutual coupling, by selecting appropriate parameters, our proposed k-times co-prime array could perform better than the nested array.

VII. Conclusion

In this paper, we propose an advanced co-prime array geometry, k-times extended co-prime array. The traditional extended co-prime array and the nested array can be generalized as the special cases of our proposed array geometry. Compared with the traditional extended co-prime array, for a given number of sensors, our proposed array configuration can offer a lager number of DoFs while reducing the array aperture. For a given SNR, it can significantly reduce the estimation error and the number of snapshots needed. On the other hand, when compared with the nested array, it performs better when the mutual coupling is taken into consideration. More importantly, it allows users to flexibly adjust (M,N) and k according to different practical application requirements.



(a) Nested array



(b) $(3,4)$ 4-ECPA

Fig. 6: MUSIC spectrum under mutual coupling

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