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Interpretation of Image Flow: Rigid Curved Surfaces in Motion

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Abstract

A new method is described for interpreting image flow (or optical flow) in a small field of view produced by a rigidly moving curved surface. The equations relating the shape and motion of the surface to the image flow are formulated. These equations are solved to obtain *explicit analytic expressions* for the motion, orientation and curvatures of the surface in terms of the spatial derivatives (up to second order) of the image flow. We state and prove some *new theoretical results concerning the existence of multiple interpretations*. Numerical examples are given for some interesting cases where multiple solutions exist. The solution method described here is simpler and more direct than previous methods. The method and the representation described here are part of a unified approach for the interpretation of image motion in a variety of cases (e.g.: planar/curved surfaces, constant/accelerated motion, etc.). Thus the representation and the method of analysis adopted here have some advantages in comparison with previous approaches.

1. Introduction

The motion of an object relative to a camera produces a moving image on the camera's image plane. The image motion thus produced contains valuable information about the three-dimensional (3D) shape and motion of the object. Recovering this 3D information from image motion is the topic of this paper. The approaches that have been taken in solving this problem fall under two major categories, *discrete* and *continuous*. In the discrete approach, the velocities of a number of distinct image feature points are used to compute the motions and relative positions of the corresponding points on the object's surface [3,19,10,17,18]. In this paper we take the continuous approach. In this approach the image motion is represented by an *image velocity field* or *image flow*. *Image flow* is a two-dimensional velocity field defined over the camera's image plane. The velocity at any point is the instantaneous velocity of the image element at that point. Some authors refer to image flow as *optical flow*. Methods for the computation of image flow from time-varying intensity images have been proposed by many researchers (e.g.: [5,4,22]). Here we are concerned with the *interpretation* of image flow, i.e. recovering the geometry and the motion of objects in a scene from the image flow.

Methods for finding the motion and orientation of a rigid *planar surface* from its image flow have been proposed by Longuet-Higgins [11], Kanatani [6], and Subbarao and Waxman [16]. For rigid *curved surfaces*, a formulation and solution method is proposed by Longuet-Higgins and Prazdny [9], but their method cannot be used if either there is no translation along the line of sight or the direction of translation, the surface normal and the line of sight all lie in a common plane. Recently Waxman, Kamgar-Parsi and Subbarao [20] have obtained a complete solution in closed-form for this problem. However some important theoretical questions concerning the multiplicity of solutions remained unanswered.

Here we reconsider the curved surface problem in the frame-work of a unified formulation for the interpretation of image flow proposed in [15]. Based on this work, we formulate the problem and derive closed-form solutions for motion, orientation and curvatures of a rigid surface in terms of its spatial image flow derivatives. In the previous approaches [9,20] the image coordinate system and the image flow derivatives are transformed corresponding to a rotation to solve the image flow equations; the solution computed is again transformed back to the original coordinate system. In contrast to this, the method described here does not involve such an intermediate step. This saves some computation. More importantly we state and prove some new theoretical results concerning the multiplicity of interpretations for image flow. Conditions for the occurrence of up to four (four being the maximum possible) solutions are stated and proved. Numerical examples are given for some interesting cases where multiple solutions exist.

Recently Maybank [12] has also obtained some results concerning the multiplicity of interpretations for image flow. His formulation uses a polar projection (spherical image screen) camera model. The main result of his work is that *an ambiguous flow field has at most three values of angular velocity compatible with it.* This should be compared to our result here which gives conditions for the occurrence of three solutions for the image flow equations. Although Maybank indicated an implicit method for constructing a case where three solutions exist, no numerical example was given. Here we give one example which is constructed using our own algorithm. According to one of our result here, if one uses *only up to second order derivatives* of image flow then there can be up to four solutions for the image flow equations. A numerical example of this case is given. In this case, as a consequence of Maybank's result, third and higher order derivatives of image flow restrict the number of possible interpretations to at most three.

In the next section we formulate the problem and derive the relevant equations. In Section 3 we give the solution in a general case, and in Section 4 we summarize the situations under which multiple solutions occur. The actual derivation of the solution and proofs regarding the occurrence and detection of multiple solutions are given in the appendices. Section 5 deals with error sensitivity analysis and numerical examples for some interesting cases where multiple solutions exist.

2. Formulation of the problem

In this section we derive a set of equations relating image flow derivatives to the shape and motion parameters of a rigid surface in motion. The derivation method in this section is primarily due to Longuet-Higgins and Prazdny [9], but it has been extended so that equations relating the image flow derivatives of any order to shape and motion parameters can be derived.

To keep the dimensionality of the parameter space small and the image flow analysis tractable, it is necessary to restrict attention to *local* analysis. Therefore, a large image is first broken into small image regions corresponding to small fields of view and the image flow is analyzed separately in each of these fields of view.

A pin-hole camera with a spherical projection screen whose center is at the *pin-hole* is a good camera model. For this camera model, due to symmetry, the image flow analysis is identical at all regions on the projection screen. However, actual camera systems usually have a planar screen. We adopt this planar projection screen model in our analysis. The geometry of the screen is entirely a matter of convenience and does not affect our results. Note that there is a one to one correspondence between an image on a curved screen such as a spherical screen and an image on a planar screen. The choice of the planar screen geometry restricts our analysis to the field of view along the optical axis. However, the image flow in a field of view not along the optical axis can be analyzed by first projecting the image velocities on to a suitable plane perpendicular to the field of view. This projection process is quite straightforward [7].

The camera model is illustrated in Figure 1. The origin of a Cartesian coordinate system *OXYZ* forms the center of projection (or ‘‘pin-hole’’) of the camera. The *Z*-axis is aligned with the optical axis and points in the direction of view. The *image plane* is at unit distance from the origin perpendicular to the optical axis. (Without loss of generality, the image plane has been taken to be *in front* of the ‘‘pin-hole’’ to avoid dealing with inverted images.) The image coordinate system *oxy* on the image plane has its origin at (0,0,1) and is aligned such that the x and y axes are, respectively, parallel to the X and Y axes.

Let the relative motion of the camera with respect to a rigid surface along the optical axis be described by translational velocity (V_X, V_Y, V_Z) and rotational velocity $(\Omega_X, \Omega_Y, \Omega_Z)$ around the origin. Due to the relative motion of the camera with respect to the surface, a 2D image flow is created by the perspective image on the image plane. At any instant of time, a point P on the surface with space coordinates (X, Y, Z) projects onto the image plane as a point p with image coordinates (x, y) given by

$$x = X / Z \quad \text{and} \quad y = Y / Z \,. \tag{2.1a,b}$$

If the position of P is given by the position vector $\boldsymbol{R}(X, Y, Z)$ then its instantaneous velocity $(\dot{X}, \dot{Y}, \dot{Z})$ is given by $\boldsymbol{U} = -(\boldsymbol{V} + \boldsymbol{\Omega} \times \boldsymbol{R})$. Therefore we have

$$\dot{X} = -V_X - \Omega_Y Z + \Omega_Z Y \,, \tag{2.2a}$$

$$\dot{Y} = -V_Y - \Omega_Z X + \Omega_X Z \,, \tag{2.2b}$$

$$\dot{Z} = -V_Z - \Omega_X Y + \Omega_Y X \,. \tag{2.2c}$$

The instantaneous image velocity of point p can be obtained by differentiating equations (2.1a,b):

$$\dot{x} = \frac{\dot{X}}{Z} - \frac{X}{Z}\frac{\dot{Z}}{Z} \quad \text{and} \quad \dot{y} = \frac{\dot{Y}}{Z} - \frac{X}{Z}\frac{\dot{Z}}{Z} \tag{2.3a,b}$$

In the above two expressions we substitute for the appropriate quantities using relations (2.2a-c,2.1a,b) to obtain

$$\dot{x} = u = \left\{ x\,\frac{V_Z}{Z} - \frac{V_X}{Z} \right\} + [\,xy\;\Omega_X - (1 + x^2)\;\Omega_Y + y\;\Omega_Z\,] \quad \text{and} \tag{2.4a}$$

$$\dot{y} = v = \left\{ y\,\frac{V_Z}{Z} - \frac{V_Y}{Z} \right\} + [\,(1 + y^2)\;\Omega_X - xy\;\Omega_Y - x\;\Omega_Z\,] \,. \tag{2.4b}$$

These equations define the instantaneous image velocity field, assigning a unique two-dimensional velocity to every point (x, y) on the surface's image.

Note that the image velocity at a point (x, y) (given by equations (2.4a,b)) in the image domain is due to the world velocity of a point (xZ, yZ, Z) in the world domain.

The value of Z is determined by the actual surface. Suppose that the visible surface is described by $Z = f(X, Y)$ in our camera-centered coordinate system; then, assuming that the surface is continuous and differentiable, a Taylor series expansion of f can be used to describe a small surface patch around the optical axis:

$$Z = Z_0 + Z_X X + Z_Y Y + \frac{1}{2} Z_{XX} X^2 + Z_{XY} X Y + \frac{1}{2} Z_{YY} Y^2 + O_3(X, Y) \tag{2.5}$$

for $Z_0 > 0$. In the above expression, Z_0 is the distance of the surface patch along the optical axis, Z_X and Z_Y are the slopes with respect to the X and Y axes, Z_{XX}, Z_{YY}, Z_{XY} are the curvatures, and the last term denotes higher order terms of the Taylor series with respect to X and Y. Using the method given in Appendix A the surface can be expressed in terms of the image coordinates of the image points:

$$Z(x,y) = Z_0 \left(1 - Z_X x - Z_Y y - \frac{1}{2} Z_{xx} x^2 - Z_{xy} xy - \frac{1}{2} Z_{yy} y^2 - O_3(x,y) \right)^{-1} \tag{2.6}$$

where

$$Z_{xx} = Z_0 \, Z_{XX} \,, \quad Z_{xy} = Z_0 \, Z_{XY} \,, \quad Z_{yy} = Z_0 \, Z_{YY} \tag{2.7a-c}$$

and $O_3(x, y)$ denotes higher order terms. Note that the curvatures are scaled by the distance along the optical axis according to relations (2.7a-c) and therefore absolute curvatures are not recoverable. The slopes Z_X, Z_Y and the scaled curvatures Z_{xx}, Z_{yy}, Z_{xy} will be collectively referred to as the *structure parameters*. Substituting for Z from equation (2.6) into equations (2.4a,b) we obtain

$$u(x,y) = \left(x\frac{V_Z}{Z_0} - \frac{V_X}{Z_0} \right) \left(1 - Z_X x - Z_Y y - \frac{1}{2} Z_{xx} x^2 - Z_{xy} xy - \frac{1}{2} Z_{yy} y^2 - O_3(x,y) \right) \tag{2.8a}$$

$$+ \left[xy\Omega_X - (1+x^2)\Omega_Y + y\Omega_Z \right],$$

$$v(x,y) = \left(y\frac{V_Z}{Z_0} - \frac{V_Y}{Z_0} \right) \left(1 - Z_X x - Z_Y y - \frac{1}{2} Z_{xx} x^2 - Z_{xy} xy - \frac{1}{2} Z_{yy} y^2 - O_3(x,y) \right) \tag{2.8b}$$

$$+\left[(1+y^2)\Omega_X - xy\Omega_Y - x\Omega_Z\right].$$

In the above equations, the distance Z_0 between the surface and the camera along the optical axis always appears in a ratio with the translational velocity $\boldsymbol{V}$ and therefore is not recoverable. Therefore, we adopt the following notation in presenting the image flow equations.

Translation parameters:

$$V_x = \frac{V_X}{Z_0}\ ,\quad V_y = \frac{V_Y}{Z_0},\quad V_z = \frac{V_Z}{Z_0}\ \text{ for } Z_0 > 0\ . \tag{2.9a-c}$$

The three components of rotation Ω_X, Ω_Y, Ω_Z and the three components of scaled translation V_x, V_y, V_z will be collectively referred to as the *motion parameters*.

Now, since the surface is assumed to be smooth (i.e. continuous and differentiable) the instantaneous image velocity in a small neighborhood around the image origin may be expressed in the form of a Taylor series:

$$u(x,y) = u_0 + u_x x + u_y y + \frac{1}{2}u_{xx}x^2 + u_{xy}xy + \frac{1}{2}u_{yy}y^2 + O_3(x,y) \text{ and} \tag{2.10a}$$

$$v(x,y) = v_0 + v_x x + v_y y + \frac{1}{2}v_{xx}x^2 + v_{xy}xy + \frac{1}{2}v_{yy}y^2 + O_3(x,y) \tag{2.10b}$$

where the subscripts indicate the corresponding partial derivatives evaluated at the image origin and $O_3(x, y)$ denotes the second and higher order terms of the Taylor series. In the above expression, the coefficients of the Taylor series expansion, u_0, v_0, u_x, . . . , etc., will henceforth be referred to as the *image flow parameters.* From the image velocity equations (2.8a,b), we can derive the following equations which relate the image flow parameters up to second order at the image origin (i.e. $x = y = 0$) to the structure and motion parameters:

$$u_0 = -V_x - \Omega_Y\ ,\qquad v_0 = -V_y + \Omega_X\ , \tag{2.11a,b}$$

$$u_x = V_z + V_x Z_X\ ,\qquad v_y = V_z + V_y Z_Y\ , \tag{2.11c,d}$$

$$u_y = \Omega_Z + V_x\, Z_Y \qquad\qquad v_x = -\,\Omega_Z + V_y\, Z_X\ , \qquad\qquad \textbf{(2.11e,f)}$$

$$u_{xx} = -\,2\, V_z\, Z_X + V_x\, Z_{xx} - 2\,\Omega_Y u_{xy} = -\,V_z\, Z_Y + V_x\, Z_{xy} + \Omega_X\ , \qquad (2.11g,h)$$

$$u_{yy} = V_x\, Z_{yy}\ , \qquad\qquad v_{xx} = V_y\, Z_{xx}\ , \qquad (2.11i,j)$$

$$v_{xy} = -\,V_z\, Z_X + V_y\, Z_{xy} - \Omega_Y \quad \text{and} \qquad v_{yy} = -2V_z Z_Y + V_y Z_{yy} + 2\Omega_X. \qquad (2.11k,l)$$

These equations, which we shall refer to as *image flow equations*, were originally derived by Longuet-Higgins and Prazdny [9]. The method we have described to derive the above equations can also be used to derive the equations relating the third and higher order image flow parameters to the structure and motion parameters. However we stop at second order as we have a sufficiently constrained system of equations (twelve equations in eleven unknowns). We will say more about this in the next section.

2.1 The nature of the image flow equations

The image flow equations (2.11a-l) form twelve non-linear algebraic equations in eleven unknowns (five structure parameters and six motion parameters). Since the system of equations is overdetermined it is found that, in general, the solution is unique, but since the equations are non-linear, in some exceptional situations multiple solutions do occur. In these equations we observe that none of the rotational components appear in the non-linear terms. All the non-linear terms are formed by the product of a structure parameter and a translation component. Further, *all* structure parameters appear only in products with the translation parameters. Therefore, it is translation through space that reveals surface structure. If there is no translation, all the structure parameters remain undetermined. Also, a curvature parameter always appears in product with a component of translation parallel to the image plane. Therefore, if there is no translation parallel to the image plane the curvatures remain undetermined by image flow parameters up to second order. There are also many situations where some of the image flow equations become dependent in which case multiple solutions occur. For example, if all the curvatures are zero (i.e. the surface is planar), or if the optical axis, direction of translation and surface normal all lie in a plane, then there are in general two solutions. A systematic analysis of all the different cases is given in the appendices. The solution in general cases and a summary of the nature of the solutions are given in the following sections.

Notice that measuring the image velocity derivatives up to second order (given by the coefficients of the Taylor series in equations (2.10a,b)) is adequate since we obtain a sufficient number of constraints (twelve) on the eleven unknowns. On the other hand spatial derivatives up to only first order are inadequate since the first six image flow equations give only six equations involving eight unknowns. Since our analysis is local, we need to solve for the structure and motion parameters separately for each small image neighborhood. However, interestingly, for planar surfaces eight image motion parameters specify the image motion field *globally* (e.g. see [16]). An intuitive explanation of this is that the motion parameters are constant everywhere in both cases, but whereas the structure parameters for a plane constituting the two slopes are constant everywhere, for curved surfaces they change.

3. Solving the image flow equations

In solving the image flow equations (2.11a-l) we use a new parameterization of the solution space; we use a trigonometric substitution which introduces two new variables r and θ which respectively correspond to the (signed) magnitude and direction of the translational component parallel to the image plane. This particular representation simplifies the task of solving the image flow equations and proving many uniqueness results. Using this representation, a unified computational approach for interpreting image flow produced by rigid surfaces has been obtained in [15].

The solutions for the image flow equations in some degenerate cases are derived in Appendix B. In the remaining part of this section we assume that we are not dealing with any of these cases. This in effect implies that we are dealing only with cases where the surface is curved (i.e. at least one of the curvature parameters is non-zero) and the translation parallel to the image plane is non-zero.

In the first two theorems of Appendix C the solution is derived for a curved surface with non-zero translation parallel to the image plane. Using the notation

$$s \equiv \sin\theta \,, \quad c \equiv \cos\theta \tag{3.1a,b}$$

$$a_1 = u_y + v_x, \quad \text{and} \quad a_2 = u_x - v_y \tag{3.2a,b}$$

the solution of equations (2.11a-l) in terms of r, θ is

$$V_x \equiv rc, \qquad V_y \equiv rs, \tag{3.3a,b}$$

$$V_z = u_x s^2 + v_y c^2 - a_1 cs, \qquad \Omega_Z = u_y s^2 - v_x c^2 + a_2 cs, \tag{3.3c,d}$$

$$Z_X = (a_1 s + a_2 c)/r, \qquad Z_Y = (a_1 c - a_2 s)/r, \tag{3.3e,f}$$

$$\Omega_X = v_0 + rs, \qquad \Omega_Y = -(u_0 + rc), \tag{3.3g,h}$$

$$Z_{xx} = \frac{1}{r}\left(u_{xx}c + v_{xx}s - 2u_0 c - 2rc^2 + 2V_z Z_X c \right) \tag{3.4a}$$

$$Z_{yy} = \frac{1}{r}\left(u_{yy}c + v_{yy}s - 2v_0 s - 2rs^2 + 2V_z Z_Y s \right) \tag{3.4b}$$

$$Z_{xy} = \frac{1}{2r}\left(s(u_{yy} + 2v_{xy} - u_{xx}) + c(v_{xx} + 2u_{xy} - v_{yy}) \right) \tag{3.4c}$$

(Note: the right hand sides of equations (3.4a,b) can be expressed in terms of only r and θ by substituting for Z_X, Z_Y and V_z from (3.3c,e,f).) In Theorem 2 of Appendix C the following three equations are derived for r and θ:

$$u_{yy} \tan^3\theta + (2u_{xy} - v_{yy}) \tan^2\theta + (u_{xx} - 2v_{xy}) \tan\theta - v_{xx} = 0 \,, \tag{3.5a}$$

$$2r^2 cs + \left(v_{xx}c - (u_{xx} - 2u_0)s \right) r - 2V_z s(a_1 s + a_2 c) = 0 \,, \tag{3.5b}$$

$$2r^2 cs + \left(u_{yy}s - (v_{yy} - 2v_0)c \right) r - 2V_z c(a_1 c - a_2 s) = 0. \tag{3.5c}$$

(Equation (3.5a) is also derived in [9,20]). Equations (3.5a-c), after substituting for V_z from (3.3c), form three equations in the two unknowns: r, θ. Therefore, to solve the image flow equations, first we solve for θ by solving the cubic equation (3.5a). Then r is obtained as the common root of the two quadratic equations (3.5b,c). If there is no common root then the corresponding solution of θ is not valid. Thus there is one extra constraint and this in general results in a unique solution. Corresponding to each solution of r and θ obtained by solving equations (3.5a-c) we get one solution for the image flow equations (from 3.3a-h,3.4a-c)

In general we can obtain an explicit solution for r by eliminating the term $2r^2cs$ from equations (3.5b,c). The solution is

$$r = \frac{2V_z\left[s(a_1s+a_2c)-c(a_1c-a_2s)\right]}{(v_{yy}-2v_0+v_{xx})c-(u_{xx}-2u_0+u_{yy})s}\ . \tag{3.6}$$

This solution should further satisfy either (3.5b) or (3.5c) in order for it to be acceptable. If either the denominator or the numerator on the right hand side is zero, then the above expression degenerates and cannot be used to solve for r. This is because we are solving for a case where the translation parallel to the image plane is finite and non-zero. In this case we have to go back to (3.5b,c) to obtain r. A complete computational algorithm for solving the image flow equations is given in Appendix C.

4. Conditions for the presence of multiple interpretations

The number of solutions for the image flow equations (2.11a-l) is equal to the number of solutions for r,θ obtained by solving equations (3.5a-c). Therefore conditions for the occurrence of multiple solutions can be obtained by an exhaustive analysis of these three equations. This analysis turns out to be a long and tedious exercise in algebra. This analysis is given in Appendix D. A summary of the analysis is given below.

Since equations (3.5a-c) form an over-constrained system of equations (three equations in two unknowns) the solution is in general unique (Theorem 9, Appendix D), but multiple solutions are possible as the equations are non-linear. Equation (3.5a) is a cubic equation involving only θ and therefore is easily solved. This gives at most three solutions for θ (in the interval $-\pi/2 < \theta \le \pi/2$). We find (in Lemma 1 of Appendix D) that for ovoid surfaces (i.e. convex or bowl shaped surfaces) only one of the three roots of equation (3.5a) is real and therefore we obtain a unique solution for θ. For cylindrical surfaces up to two, and for saddle shaped surfaces up to three solutions for θ are possible (Lemma 1 of Appendix D). Note that, for a given θ we can immediately solve for V_z and Ω_Z from relations 3.3c,d. Therefore, given θ, equations (3.5b) and (3.5c) reduce to simple quadratic equations in r and therefore can be easily solved. The common root(s) of these two equations is (are) the solution(s) for r. If there is no common root then the given solution for θ is not acceptable. If the roots of the two quadratic equations obtained from equations (3.5b,c) are identical (which is the case when the coefficients of the two equations are proportional) then we have two solutions for a given θ. This is found to be the case (Theorems 7 and 8 in Appendix D) when the direction of translation, the surface normal and the optical axis all lie in a common plane. However, in this case, the solution for r becomes unique when the two roots of the quadratic equation (3.5b or 3.5c) are equal. In one case (Theorem 8, Appendix D) the two roots become equal when the direction of translation is parallel to the surface normal.

There are two special situations when equations (3.5b,c) reduce to a single linear equation in r (as compared to two quadratic equations in a general case) for a given θ : (i) a specular saddle surface (i.e. the tangent plane to the surface is frontal or parallel to the image plane and the surface itself is saddle shaped) with mean scaled curvature equal to -1 (Theorem 5 of Appendix D), and (ii) a saddle or a cylindrical surface with mean scaled curvature equal to -1, no translation along the optical axis, and the slopes and curvatures satisfy a certain condition (D17) (Theorem 6 of Appendix D). In these situations all solutions for θ obtained by solving equation (3.5a) are acceptable (Theorem 3 of Appendix D) and each of these results in one solution for the image flow equations. In fact it is found that for case (i) above three solutions exist (Theorem 5 of Appendix D) and for case (ii) up to two solutions are possible (Theorem 6 in Appendix D).

There is one case where there are two solutions for θ and for each of these there are two solutions for r, thus leading to a total of four solutions. This occurs when there is a rare coincidence (relations D6c,D23) of the translation vector, slopes, and curvatures (see

Theorem 7, Appendix D). However the presence of four solutions is an artifact of using only up to second order image flow parameters. Use of higher order flow parameters should prevent any more than three solutions according to Maybank [12]. A more precise account of the nature of the solution in various cases follows.

4.1 Overall summary of the nature of the solutions

This subsection summarizes the detailed results presented in Appendices B, and D. The most important results of these appendices are the explicit statement and proofs of all the conditions for the occurrence of up to four solutions, four being the maximum possible. Following is a summary of the nature of the solutions to the image flow equations (2.11a-l) in different cases. Here we list the different cases in a sequential order such that the occurrence of a particular case is detected by the absence of the previous cases and checking for the satisfaction of one or more constraints on the image flow parameters.

(i) If there is no translation, the structure parameters are undetermined and the solution is unique.

(ii) If either there is no translation parallel to the image plane or the surface is planar and frontal, then there can be up to three solutions for the image flow equations.

(iii) If the surface is planar then there are two solutions with the exception of the following cases:

(a) There is no translation along the optical axis.

(b) Translation is parallel to the surface normal.

(iv) There are three solutions if the surface is a specular saddle and the mean scaled curvature is −1 (i.e. $Z_{xy}^2 - Z_{xx}Z_{yy} > 0$, $Z_X = Z_Y = 0$ and $(Z_{xx}+Z_{yy})/2 = -1$.).

(v) There are two solutions if the surface is saddle or cylindrical (i.e. $Z_{xy}^2 - Z_{xx}Z_{yy} \geq 0$), there is no translation along the optical axis and the mean scaled curvature is −1 except when the translation vector, surface normal and the optical axis all lie in a plane; in this case the solution is unique.

(vi) There can be up to four solutions if the translation vector, surface normal and the optical axis all lie in a plane.

(vii) In cases other than the ones listed above the solution is unique.

5. Error sensitivity and numerical examples

In the compuatational approach described here, the solution for the image flow equations is given by *explicit analytic expressions.* Therefore, a theoretical sensitivity analysis can be done by taking *error differentials.* If the uncertainty in the input image flow parameters are known, then approximate bounds on the maximum error in the 3D structure and motion parameters can be estimated. This analysis holds for *all* cases. In contrast, the sensitivity analyses of previous approaches are based on a few numerical examples; a general analysis was not possible as closed-form solutions were not available [1,2,21,22].

The error in the output 3D structure and motion parameters depend on the uncertainty in the input flow parameters. The errors in the flow parameters in turn depend on two factors: the image quality (in terms of spatial resolution, gray level resolution, and the noise level in grey level and pixel registration) and the computational method employed in estimating the flow derivatives. If a tolerance is specified for the output structure and motion, then it is in principle possible to obtain an approximate idea of the required image quality.

A sensitivity analysis based on error differentials gives only the worst case behavior. Therefore such an analysis is often inadequate in practical applications. A more accurate analysis is difficult unless a domain of application is specified. This difficulty arises from the non-linear nature of the problem.

5.1 Estimation of maximum absolute error

The maximum absolute error in the computation of an analytic function can be estimated using the total differential of the function (cf. [14]). Let $y = f(x_1, x_2, \ldots, x_n)$ be an analytic function and $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ be the errors in the corresponding arguments. Then, for sufficiently small absolute values of $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, the error Δy in y can be shown to satisfy the relation

$$|\Delta y| \leq \left|\frac{\partial f}{\partial x_1}\right||\Delta x_1| + \left|\frac{\partial f}{\partial x_2}\right||\Delta x_2| + \ldots + \left|\frac{\partial f}{\partial x_n}\right||\Delta x_n| \tag{5.1}$$

Relation (5.1) can be used to estimate the maximum absolute errors in the scene parameters given the uncertainties in the image parameters. For example, consider the estimation of error in $\tan\theta$ by solving the cubic equation (3.5a). Let the cubic equation be represented by

$$x_1 + x_2 y + x_3 y^2 + x_4 y^3 = 0 \,. \tag{5.2}$$

Then, using relation (5.1) we can obtain

$$|\Delta y| \leq \frac{|\Delta x_1| + |\Delta x_2||y| + |\Delta x_3||y^2| + |\Delta x_4||y^3|}{|x_2 + 2x_3 y + 3x_4 y^2|}. \tag{5.3}$$

Therefore, if the uncertainties in the coefficients of the cubic equation are given, then the uncertainty in the roots can be estimated. Errors in the other unknowns can be estimated similarly.

The actual error is usually much smaller than that given by relation (5.1). We give here the result of an experiment where the correct solution and the solution obtained from noisy input are given. This example is included to give an idea about the sensitivity of the approach. Bounds on the error are not estimated.

Example: The estimated image flow parameters for a curved surface in rigid motion using the *velocity functional method* (Waxman and Wohn, [22]) are given (from Wohn [23]). The estimation was based on a noisy (5%) normal velocity field along a contour. The contour spanned approximately a ten degree field of view. The result of solving the image flow equations in this case is given below.

Noisy input image flow parameters:

u_0 :	-6.017523	u_x :	2.999429	u_y :	-0.088125
u_{xx} :	3.009457	u_{xy} :	0.023188	u_{yy} :	3.052289
v_0 :	-3.965109	v_x :	0.087267	v_y :	2.997560
v_{xx} :	2.020104	v_{xy} :	-0.015687	v_{yy} :	2.046979

Solution of image flow equations:

(V_x, V_y, V_z) :	(5.991399 3.984022 2.998529)
(O_x, O_y, O_z) :	(0.018913 0.026124 -0.087696)
(Z_X, Z_Y) :	(-0.000008 -0.000005)
(Z_{xx}, Z_{yy}, Z_{xy}) :	(0.506775 0.001294 0.509323)

Original scene parameters:

Z_0 :	1.000000
(V_x, V_y, V_z) :	(6.000000 4.000000 3.000000)
(O_x, O_y, O_z) :	(0.034966 0.017453 -0.087266)
(Z_X, Z_Y) :	(0.000000 0.000000)
(Z_{xx}, Z_{yy}, Z_{xy}) :	(0.500000 0.00000 0.500000)

Here we see that that the computed solution compares well with the original values.

In general it has been observed that the error in the input image flow parameters recovered by Wohn [23] are large for curved surfaces, perhaps due to the inadequate spatial resolution of the images. Consequently, the error in the output 3D parameters are large.

5.2 Numerical examples of multiple solution cases

The solution method described here was implemented on a Symbolics 3600 computer. Random values of motion and structure parameters were generated and the image flow parameters were computed using relations (2.11a-l). These image flow parameters were given as input to a program to solve the image flow equations. The program was successfully run on hundreds of test examples. Many of these examples were specifically designed (using the results in Appendix B and D) to produce the cases of multiple interpretations. Some non-trivial examples of these cases are given here. In each case the solutions of (θ, r) are given. For each (θ, r), the corresponding solution for the structure and motion can be obtained using relations (3.3a-h,3.4a-c). The validity of the solution thus obtained can be easily verified by computing the image flow parameters using relations (2.11a-l) and comparing them to the input flow parameters. More details on numerical examples can be found in [15]. (All values are rounded to the sixth decimal place.)

Example 1 : For a curved surface with non-zero translation parallel to the image plane, there are three solutions if the surface is a specular saddle and the mean curvature is -1.

Input image motion parameters:

u_0 : 9.560000	v_0 : 13.570000	u_x : -9.140000	v_x : 8.960000
u_y : -8.960000	v_y : -9.140000	u_{xx} : 14.563000	v_{xx} : -3.402000
u_{xy} : -5.825180	v_{xy} : -40.404280	u_{yy} : 4.557000	v_{yy} : 30.542000

The set of solutions for (θ, r):

{ (-0.035108 , -50.740273) , (1.381851 , -10.399291) , (1.329556 , -7.785441) }.

Example 2 : For a curved surface with non-zero translation, there are two solutions if it is saddle or cylindrical, there is no translation along the optical axis, its mean scaled curvature is -1 and the slopes and curvatures are related by relation (D17).

Input image motion parameters:

u_0 : -13.090000	v_0 : 10.130000	u_x : -2.025000	v_x : 6.870000
u_y : -0.618000	v_y : -3.050400	u_{xx} : 2.625994	v_{xx} : -12.576974
u_{xy} : -23.874837	v_{xy} : 10.120568	u_{yy} : -28.805994	v_{yy} : 32.836974

The set of solutions for (θ, r):

{ (-1.187512 , 31.733480) , (-0.545848 , 4.738576) }

Example 3 : There are four solutions when the vector given by (slope along the X-axis, slope along the Y-axis, 1+mean scaled curvature) is parallel to the translation vector.

Input image motion parameters :

u_0 : -8.300000	v_0 : 2.530000	u_x : 8.886919	v_x : -4.112594
u_y : -1.332594	v_y : 3.122606	u_{xx} : -30.694250	v_{xx} : 3.251107
u_{xy} : -12.390270	v_{xy} : -4.858356	u_{yy} : 14.094250	v_{yy} : 1.808893

The set of solutions for (θ, r):

{ (-0.378468 , 2.733441) , (-0.378468 , -5.917901)

(1.192328 , -15.826524) , (1.192328 , -4.995007) }.

6. Conclusion

We have described a new method for interpreting image flow produced by a rigidly moving curved surface. We have also stated and proved conditions for the presence of multiple interpretations. Previous work along with the work reported here leads us to believe that the (local) analysis of instantaneous image flow produced by rigidly moving surfaces is now well understood. We now know how to formulate the image flow equations, how to solve them, and what are the situations for which multiple interpretations exist. However, practical applications of this flow analysis requires very high quality images (in terms of spatial and gray level resolution) and large computational power unless severe restrictions are imposed on the shape and motion of surfaces in the scene (e.g.: planarity of surfaces, pure translatory motion with no rotation, etc). This is evident as second order image flow derivatives are needed to recover the surface structure. One way to deal with this problem is not to restrict the analysis to instantaneous flow, but use temporal variation of the flow as well. A systematic method to incorporate temporal information in image flow analysis is described in [15]. It is found that *only first order spatial and temporal derivatives of image flow* are sufficient to recover the structure and rigid motion of a surface. Further, the image flow problem has been formulated for arbitrary shapes and transformations of surfaces (e.g. non-rigid, non-uniform motion, etc.). Now we are focusing our efforts on obtaining robust computational techniques for image flow analysis.

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APPENDIX A. Expressing a surface in terms of image coordinates

Here we give a method of deriving the function that maps image points (x,y) on the image plane to points on the surface in the scene along the optical axis. Since the image at a point (x, y) on the image plane corresponds to the point (xZ, yZ, Z) in the scene, our goal is to express Z in terms of (x, y) and the surface structure parameters. In [9,21] Z was so expressed only up to second order terms of (x, y). Below we give a systematic method which can be used to express Z up to any desired order of terms in (x, y).

Assuming that the surface is smooth and is given by $Z = f(X,Y)$ we can expand $f(X,Y)$ in a Taylor series:

$$Z = a_0+a_1X+a_2Y+a_3X^2+a_4XY+a_5Y^2+a_6X^3+..... \tag{A1}$$

Using equations (2.1a,b) and equation (A1) we can obtain an implicit expression for Z in terms of the image coordinates x, y :

$$Z = a_0 + Z\,(a_1 x + a_2 y + \underline{Z}\,(a_3 x^2 + a_4 xy + a_5 y^2 + Z\,(a_6 x^3 + ...)))$$ (A2)

Now we systematically substitute for the appropriate *Z*s on the right hand side to eliminate second and higher order terms in *Z* on the right hand side of equation (A2). Substituting the entire right hand side of equation (A2) for the *Z* underlined in equation (A2) we get

$$Z = a_0 + Z\,(a_1 x + a_2 y + (a_0 + Z\,(a_1 x + a_2 y + ...))$$ (A3)

$$(a_3 x^2 + a_4 xy + a_5 y^2 + Z\,(a_6 x^3 + ...)))$$

Rearranging terms in equation (A3) we have

$$Z = a_0 + Z(a_1 x + a_2 y + a_0 a_3 x^2 + a_0 a_4 xy + a_0 a_5 y^2 + a_0 \underline{Z}(a_6 x^3 + ...)$$ (A4)

$$+\underline{Z}(a_1 x + a_2 y + ...)(a_3 x^2 + a_4 xy + a_5 y^2 + Z(a_6 x^3 + ...)))$$

We again substitute for the *Z*s underlined in equation (A4) the entire expression on the right hand side of equation (A4):

$$Z = a_0 + Z(a_1 x + a_2 y + a_0 a_3 x^2 + a_0 a_4 xy + a_0 a_5 y^2 + (a_0^2 a_6 + a_0 a_1 a_3) x^3 + ...).$$ (A5)

Continuing this recursive substitution procedure, *Z* can be expressed explicitly in terms of the image coordinates *x*, *y* to any required order of terms. Using $O_3(x,y)$ to denote third and higher order terms equation (A5) can be written as

$$Z = a_0 + Z\,(a_1 x + a_2 y + a_0 a_3 x^2 + a_0 a_4 xy + a_0 a_5 y^2 + O_3(x,\ y))\ .$$ (A6)

Rearranging terms in equation (A6) we get

$$a_0 = Z\,(1 - a_1 x - a_2 y - a_0 a_3 x^2 - a_0 a_4 xy - a_0 a_5 y^2 - O_3(x,y))\ .$$ (A7)

Equation (A7) can be used to obtain an expression for *Z* which is explicit up to second order terms:

$$Z = a_0\,[1 - a_1 x - a_2 y - a_0 a_3 x^2 - a_0 a_4 xy - a_0 a_5 y^2 - O_3(x,y)]^{-1}\ .$$ (A8)

APPENDIX B. Some degenerate cases and conditions

In this appendix we mainly consider some degenerate cases. Conditions for the presence of these cases are given in terms of the image flow parameters and in each case the solution of the image flow equations is given. This enables us to analyze a general case by precluding the occurrence of these degenerate cases. This strategy of analysis is almost a necessity due to the non-linear nature of the image flow equations.

Below we systematically consider the different cases in a sequential order. The solution is presented in a sequence of theorems and lemmas. Typically, a theorem or a lemma gives the solution when a specified condition is true. The condition part usually specifies that the condition for none of the preceding theorems or lemmas is true but a certain condition specific to this (theorem/lemma) holds. This style of presentation suggests an implementation algorithm based on testing for conditions on the image flow parameters. Whenever the proof of a theorem or a lemma is obvious or simple we have not included the proof here.

B1. Some degenerate cases

Theorem 1 : The condition

$$u_x=v_y=u_y+v_x=u_{xx}-2u_0=u_0-v_{xy}=v_{yy}-2v_0=v_0-u_{xy}=u_{yy}=v_{xx}=0. \qquad \text{(B1)}$$

is true *if and only if* there is no translation, i.e. $V_x=V_y=V_z=0$.

Lemma 1a : Under the condition stated in the above theorem the solution of the image flow equations is

$$Z_X,\ Z_Y,\ Z_{xx},\ Z_{yy} \text{ and } Z_{xy} \text{ are indeterminate}, \qquad \text{(B2a)}$$

$$(V_x,\ V_y,\ V_z) = (0,\ 0,\ 0)\ , \quad \text{and} \qquad \text{(B2b)}$$

$$(\Omega_X,\ \Omega_Y,\ \Omega_Z) = (v_0,\ -u_0,\ u_y)\ . \qquad \text{(B2c)}$$

Theorem 2 : The condition under Theorem 1 is false and

$$(u_x - v_y = u_y + v_x = u_{xx} - 2v_{xy} = v_{yy} - 2u_{xy} = u_{yy} = v_{xx} = 0) \tag{B3}$$

if and only if one of the following is true:

(i) there is no translation parallel to the image plane, i.e.

$$V_x = V_y = 0 \tag{B4a}$$

(ii) the surface is frontal, planar, and V_x=0, i.e.

$$Z_X = Z_Y = Z_{xx} = Z_{yy} = Z_{xy} = V_x = 0\ . \tag{B4b}$$

(iii) the surface is frontal, planar, and V_y=0, i.e.

$$Z_X = Z_Y = Z_{xx} = Z_{yy} = Z_{xy} = V_y = 0\ . \tag{B4c}$$

Lemma 2a : Under the conditions of the above theorem there can be up to three solutions for the image flow equations, one solution each for the three cases (B4a),(B4b), and (B4c). The solutions are

$$(i)\ \ Z_{xx},\ Z_{yy}\ \text{ and }\ Z_{xy}\ \text{ are indeterminate ,} \tag{B5a}$$

$$(V_x,\ V_y,\ V_z) = (0,\ 0,\ u_x)\ , \tag{B5b}$$

$$(\Omega_X,\ \Omega_Y,\ \Omega_Z) = (v_0,\ -u_0,\ u_y)\ , \tag{B5c}$$

$$(Z_X,\ Z_Y) = \left[\ (u_0 - v_{xy})/u_x,\ (v_0 - u_{xy})/u_x \right] \tag{B5d}$$

$$(ii)\ \ Z_X = Z_Y = Z_{xx} = Z_{yy} = Z_{xy} = 0\ , \tag{B6a}$$

$$(\ V_x,\ V_y,\ V_z) = (0,\ u_{xy} - v_0,\ u_x)\ , \tag{B6b}$$

$$(\Omega_X,\ \Omega_Y,\ \Omega_Z) = (u_{xy},\ -v_{xy},\ u_y)\ , \tag{B6c}$$

and

$$(ii)\ \ Z_X = Z_Y = Z_{xx} = Z_{yy} = Z_{xy} = 0\ , \tag{B7a}$$

$$(V_x,\ V_y,\ V_z) = (v_{xy} - u_0,\ 0,\ u_x)\ , \tag{B7b}$$

$$(\Omega_X,\ \Omega_Y,\ \Omega_Z) = (u_{xy},\ -v_{xy},\ u_y)\ . \tag{B7c}$$

B2. Condition for the moving surface to be planar

Theorem 3 : Conditions under Theorem 1 and Theorem 2 are both false and

$$(u_{yy}=v_{xx}=u_{xx}-2v_{xy}=v_{yy}-2u_{xy}=0) \tag{B8}$$

if and only if the surface is planar, and there is a finite translation parallel to the image plane, i.e.

$$Z_{xx} = Z_{yy} = Z_{xy} = 0\ , \tag{B9a}$$

$$V_x \neq 0 \quad \text{OR} \quad V_y \neq 0\ . \tag{B9b}$$

This case of planar surfaces is treated elsewhere (e.g.: [16]).

B3. Condition for the moving surface to be curved

Theorem 4 : If conditions under Theorem 1, Theorem 2, and Theorem 3 are all false, then the surface is curved (i.e. at least one of the curvatures is non-zero) and translation parallel to the image plane is non-zero; i.e.

$$(Z_{xx}\neq 0 \text{ OR } Z_{yy}\neq 0 \text{ OR } Z_{xy}\neq 0)\ \ \text{AND} \tag{B10a}$$

$$V_x \neq 0 \quad \text{OR} \quad V_y \neq 0\ . \tag{B10b}$$

In the following discussion the condition part of the above theorem (i.e. conditions under Theorem 1, Theorem 2, and Theorem 3 are all false) will be referred to as the *curved-surface- condition.*

APPENDIX C. Solving the image flow equations

Throughout this appendix we assume that the surface in motion is curved (i.e. at least one of the curvatures is non-zero), and translation parallel to the image plane is not zero. Conditions for the presence of this case in terms of the image flow parameters is given in Theorem 4 of Appendix B.

We give here a method to solve the image flow equations by first solving for θ, then for r, and then for the other unknowns.

C1. Solution for orientation and motion in terms of r and θ

Theorem 0: Suppose that translation parallel to the image plane is not zero and let r and θ be such that

$$V_x \equiv r\cos\theta \text{ and } V_y \equiv r\sin\theta \text{ for } -\pi/2 < \theta \le \pi/2 . \qquad \text{(C1a,b)}$$

Then, using the notation

$$s \equiv \sin\theta \text{ and } c \equiv \cos\theta , \qquad \text{(C2a,b)}$$

the motion and orientation are given by relations (3.3a-h).

Proof : Relations (3.3a,b,g,h) are easily obtained from relations (2.11a,b) and (C1a,b). From relations (3.2a,b), (2.11c-f), and (C1a,b) we can get

$$a_1 = rcZ_Y + rsZ_X \text{ and } a_2 = rcZ_X - rsZ_Y . \qquad \text{(C3a,b)}$$

Solving for Z_X and Z_Y from above equations, we get relations (3.3e,f). Now, from relations (2.11c), (C1a), and (3.3e) we can get

$$V_z = u_x - a_1 cs - a_2 c^2 . \qquad \text{(C4a)}$$

Or, using relation (3.2b) and the identity $s^2+c^2=1$,

$$V_z = u_x(s^2+c^2) - a_1 cs - a_2 c^2 . \qquad \text{(C4b)}$$

Relation (3.3c) can be obtained from the above relation. The derivation of relation (3.3d) is similar to that of relation (3.3c) •

C2. Solution for curvatures

Theorem 1: Suppose that the curved-surface-condition is true. Then the solution(s) of the curvatures are given in terms of θ and r by relations (3.4a-c).

Proof : In equation (2.11g) we substitute for V_x and Ω_Y from relations (3.3a,h), multiply the resulting equation by c and rearrange terms to get

$$rc^2\, Z_{xx} = u_{xx}c - 2u_0 c - 2rc^2 + 2V_z Z_X c. \tag{C6a}$$

In equation (2.11j) we substitute for V_y from (3.3b) and multiply the resulting equation by s to get

$$rs^2 Z_{xx} = v_{xx}s\ . \tag{C6b}$$

Adding equations (C6a,b) and simplifying we can get equation (3.4a).

Derivation of equation (3.4b) is similar to that of (3.4a) above. In this case we multiply (2.11l) by s, (2.11i) by c, add the resulting equations, do appropriate substitutions, and simplify to get (3.4b).

From equations (2.11g,k,3.3a,b) we get

$$u_{xx} - 2v_{xy} = rcZ_{xx} - 2rsZ_{xy} \tag{C7a}$$

From equations (2.11j,3.3b) we have

$$Z_{xx}s = v_{xx}/r. \tag{C7b}$$

Multiplying (C7a) by s and using (C7b) we get

$$s(u_{xx} - 2v_{xy}) = cv_{xx} - 2rs^2 Z_{xy}. \tag{C7c}$$

Now we use equations (2.11l,h,3.3a,b) to get

$$v_{yy} - 2u_{xy} = rsZ_{yy} - 2rcZ_{xy}. \tag{C8a}$$

From equations (2.11i,3.3a) we have

$$Z_{yy}c = u_{yy}/r. \tag{C8b}$$

Multiplying (C8a) by c and using (C8b) we get

$$c(v_{xx} - 2u_{xy}) = su_{yy} - 2rc^2 Z_{xy}. \tag{C8c}$$

Adding (C7c,C8c) and simplifying we get (3.4c) ●

C3. Constraints on r and θ

Theorem 2 : When the curved-surface-condition is true, the parameters r and θ are related to the image flow parameters by the relations (3.5a-c).

Proof : In equation (C7c) we substitute for Z_{xy} from (3.4c) and simplify to get

$$s(u_{xx}-2v_{xy}) = v_{xx}c + s^3(u_{xx}-2v_{xy}) - s^3 u_{yy} + s^2 c(v_{yy}-2u_{xy}) - s^2 c v_{xx}. \qquad \text{(C9a)}$$

Using the identity $s^2+c^2 = 1$, this can be rewritten as

$$s(u_{xx}-2v_{xy}) = v_{xx}c + (1-c^2)s(u_{xx}-2v_{xy}) - s^3 u_{yy} \qquad \text{(C9b)}$$

$$+ s^2 c(v_{yy}-2u_{xy}) - (1-c^2)c v_{xx}.$$

Simplifying the above equation and dividing by c^3 we get (3.5a).

In equation (2.11g) we substitute for V_x and Ω_Y from relations (3.3a,h), multiply the resulting equation by s and rearrange terms to get

$$rcs\ Z_{xx} = u_{xx}s - 2u_0 s - 2rcs + 2V_z Z_X s. \qquad \text{(C10a)}$$

In equation (2.11j) we substitute for V_y from (3.3b) and multiply the resulting equation by c to get

$$rcsZ_{xx} = v_{xx}c\ . \qquad \text{(C10b)}$$

Subtract (C10b) from (C10a), substitute for Z_X from (3.3e), and simplify to obtain (3.5b).

Derivation of equation (3.5c) is similar to that of (3.5b) above. In this case we multiply (2.11l) by c, (2.11i) by s, subtract the resulting equations, do appropriate substitutions, and simplify to get (3.5c) ●

C4. Summary of computational algorithm

First check if the input image flow corresponds to a degenerate case as discussed in Appendix B, and if so, obtain the solution according to Appendix B. Otherwise first solve for θ by solving the cubic equation (3.5a) and then solve for r by simultaneously solving equations (3.5b,c). Now, for each set of solutions obtained for θ, r, solve for the other unknowns from relations (3.3a-h,3.4a-c).

If the input image flow parameters are in error due to noise, then there may be no solution for θ and r which satisfies all three equations (3.5a-c). In this case we can solve for θ and r by using only two of these three equations. If this gives multiple solutions then we select the one which is closest to satisfying the third equation. In this solution method the solution we obtain could be complex valued. For example the solution for r obtained by solving a quadratic equation may have a small imaginary part. In this case we may either ignore the imaginary part or take the magnitude to be the solution. In general it will be necessary to use some heuristics to deal with noisy input data.

A note on solving polynomial equations in $\tan\theta$*:* while solving a cubic, quadric, or a linear equation in $\tan\theta$, the coefficient of the highest power of $\tan\theta$ may be zero. In this case we can take one of the solutions for θ to be $\pi/2$ and then proceed to solve the next lower order equation in $\tan\theta$. This gives correct results because we have assumed $-\pi/2 < \theta \leq \pi/2$.

APPENDIX D. Conditions for multiple interpretations

In this section we give conditions for the presence of multiple interpretations for the structure and motion of a moving curved surface. Throughout this section we assume that we are dealing with a curved surface with non-zero translation parallel to the image plane. The condition for this case in terms of the image flow parameters is given in Theorem 4 of Appendix B.

The number of solutions to the image flow equations is equal to the number of solutions for r and θ obtained by solving equations (3.5a-c). Therefore we analyze equations (3.5a-c) exhaustively to derive the conditions for multiple interpretations.

D1. Solutions for θ

The following two lemmas are related to the nature of the solution for θ as determined by equation (3.5a). The results of these lemmas were known earlier [20].

Theorem 1 : The roots of the cubic equation (3.5a) are

$$\tan\theta = \frac{V_y}{V_x}\ ,\ \frac{1}{Z_{yy}}\,(-Z_{xy} \pm \sqrt{Z_{xy}^2 - Z_{xx}Z_{yy}})\ . \qquad \text{(D1)}$$

Proof : The coefficients of the cubic equation (3.5a) can be respectively expressed in terms of the structure and motion parameters as

$$V_x Z_{yy}\ ,\ \ 2V_x Z_{xy} - V_y Z_{yy}\ ,\ \ V_x Z_{xx} - 2V_y Z_{xy}\ ,\ \ -V_y Z_{xx}\ . \qquad \text{(D2)}$$

Therefore, equation (3.5a) can be factored as

$$(V_x \tan\theta - V_y)\,(Z_{yy}\tan^2\theta + 2Z_{xy}\tan\theta + Z_{xx}) = 0\ . \qquad \text{(D3)}$$

The roots of the above equation are given by (D1) ●

Lemma 1 : There is one real solution for θ if the surface is ovoid (i.e. bowl shaped), there are two real solutions if the surface is cylindrical and there are three real solutions if the surface is a saddle.

Proof : The shape of a surface can be inferred from the sign of the *Gaussian curvature* (cf. [13]). The Gaussian curvature has the same sign as the expression

$$Z_{xx}Z_{yy} - Z^2{}_{xy}\ . \qquad \text{(D4)}$$

If the sign is positive then the surface is ovoid, if it is zero then it is a cylinder (or a plane) and if it is negative then it is a saddle. From this and the expressions for the solution of θ given by (D1) the proof is clear ●

D2. Multiple interpretations due to multiple solutions for θ

Although equation (3.5a) may give up to three solutions for θ, in most cases only one of them corresponding to the correct physical interpretation satisfies equations (3.5b,c) for some real value of r. In the following we consider cases where θ has two or more solutions. Theorem 2 to Lemma 3d correspond to this case.

D2.1 A necessary condition for multiple solutions to θ

Theorem 2: For a given set of image flow parameters, the curved-surface-condition is true, and

$$(v_{yy} - 2v_0 + v_{xx} = 0) \text{ and} \tag{D5a}$$

$$(u_{xx} - 2u_0 + u_{yy} = 0) \tag{D5b}$$

if and only if one of the following is true :

$$(i) \quad (Z_X = Z_Y = 0) \text{ and } (Z_{xx} + Z_{yy} + 2 = 0) \tag{D6a}$$

$$(ii) \quad (V_z = 0) \text{ and } (Z_{xx} + Z_{yy} + 2 = 0) \tag{D6b}$$

$$(iii) \quad (V_z \neq 0) \text{ and } (Z_{xx} + Z_{yy} + 2 \neq 0) \text{ and} \tag{D6c}$$

$$\frac{V_x}{Z_X} = \frac{V_y}{Z_Y} = \frac{V_z}{1 + (Z_{xx} + Z_{yy})/2}$$

(Note: Case (i) implies that the surface is frontal (or specular) and the mean scaled curvature is –1; Case (ii) implies that there is no translation along the optical axis and the mean scaled curvature is –1; Case (iii) implies that the direction of translation is parallel to the vector: $(Z_X, Z_Y, 1 + (Z_{xx} + Z_{yy})/2.)$

Proof : From equations (2.11a,b,g,i,j,l) we have

$$v_{yy} - 2v_0 + v_{xx} = V_y(Z_{xx} + Z_{yy} + 2) - 2V_z Z_Y \text{ and} \tag{D7a}$$

$$u_{xx} - 2u_0 + u_{yy} = V_x(Z_{xx} + Z_{yy} + 2) - 2V_z Z_X \,. \tag{D7b}$$

Therefore, conditions (D5a,b) hold if and only if

$$V_y(Z_{xx}+Z_{yy}+2) = 2V_z Z_Y \text{ and} \tag{D8a}$$

$$V_x(Z_{xx}+Z_{yy}+2) = 2V_z Z_X \text{ .} \tag{D8b}$$

Now recall that $V_x \neq 0$ or $V_y \neq 0$ since $r \neq 0$. Now consider the logical expression

$(D\,8a)$ and $(D\,8b)$ and $\left[(Z_{xx}+Z_{yy}+2=0) \; or \; (Z_{xx}+Z_{yy}+2\neq 0)\right]$ and

$$(V_x\neq 0 \; or \; V_y\neq 0) \text{ and } (V_z=0 \; or \; V_z\neq 0). \tag{D9}$$

Expanding the above logical expression and simplifying, we get the disjunction of the three clauses (D6a-c). ●

Theorem 3 : Under the conditions of Theorem 2 when either (D6a) or (D6b) is true equations (3.5b,c) are independent of θ, i.e. they cannot be used to solve for θ (nor do they give rise to any constraint on θ).

Proof : For the cases (D6a,b) we get from equations (3.5b,c) the following constraint on θ :

$$(v_{yy}-2v_0+v_{xx}) \cos\theta = (u_{xx}-2u_0+u_{yy}) \sin\theta \text{ .} \tag{D10}$$

The above constraint is identically true for all values of θ under the conditions (D5a,b). ●

If conditions of Theorem 2 are true then we can determine which one of the three conditions (D6a-c) is actually true in that order by further testing the image flow parameters.

Theorem 4 : Under the conditions of Theorem 2,

(i) case (D6a) is true if and only if $a_1 = a_2 = 0$,

(ii) case (D6a) is false and case (D6b) is true if and only if

$$\tan\theta = \frac{a_1 \pm \sqrt{a_1^2 - 4u_x v_y}}{2u_x} \text{ .} \tag{D11}$$

(iii) cases (D6a,b) are false and (D6c) is true if and only if

$$\tan 2\theta = a_1 / a_2 \ . \tag{D12}$$

Proof : Case (i) is obvious from equations (3.3e,f). Case (ii) is easily proved by equating the expression (3.3c) for V_z to zero. To prove the last case, we solve for θ from equations (3.5b,c) as follows. Eliminating the term $2r^2cs$ from the expressions (3.5b,c) and rearranging terms, we get the following constraint on θ :

$$r\left[(v_{yy}-2v_0+v_{xx})c-(u_{xx}-2u_0+u_{yy})s\right] \tag{D14}$$

$$= 2V_z\left[s(a_1s+a_2c)-c(a_1c-a_2s)\right] \ .$$

Under the conditions (D5a,b), the left hand side of equation (D14) is identically zero. Therefore equating the right hand side to zero we get equation (D12).●

D2.2 Triple solution theorem

Theorem 5 : Under the conditions of Theorem 2, equations (2.11a-l) can have three solutions *if and only if* case (D6a) is true (i.e. the surface is frontal and the mean scaled curvature is -1) and the surface is a saddle, i.e.

$$Z^2_{xy} - Z_{xx}Z_{yy} > 0 \ . \tag{D15}$$

Proof : In this case, all the values of θ computed by solving equation (3.5a) are valid since there is no other extra constraint on θ (from Theorem 4). Under condition (D15) there are three solutions for θ given by relations (D1). For each value of θ this case results in a unique value for *r* obtained by solving (3.5b) or (3.5c) given by

$$r = -\frac{1}{2}\left\{\frac{v_{xx}}{s}+\frac{u_{yy}}{c}\right\} . \tag{D16}$$

Therefore, there are three solutions in this case. ●

D2.3 Double solution theorem 1

Theorem 6 : Under the conditions of Theorem 2 suppose that case (D6a) is false. Then equations (2.11a-l) can have up to two solutions if case (D6b) is true (i.e. translation along the optical axis is zero and the mean scaled curvature is -1) and the two solutions of equation (D11) are also the roots of the the cubic equation (3.5a).

Proof : In this case equations (3.5a) and (D11) are the only constraints on θ. For each solution common to these equations we get a unique value for r obtained by solving equations (3.5b,c) given by relation (D16). Therefore there can be up to two solutions for the image flow equations (2.11a-l). •

Lemma 2 : Under the conditions stated in Theorem 6 the structure parameters of the surface satisfy (D6b and) the following constraint:

$$\frac{Z_Y}{Z_X} = \frac{1}{Z_{yy}} \left(-Z_{xy} \pm \sqrt{Z_{xy}^2 - Z_{xx} Z_{yy}}\right) . \tag{D17}$$

Proof : Substituting for terms on the right hand side of equation (D11) in terms of the structure and motion parameters (from relations (2.11c-f) and (3.2a)) we can show that the two solutions for θ are

$$\tan\theta = \frac{V_y}{V_x} \ , \ \frac{Z_Y}{Z_X} \ . \tag{D18}$$

Since these are also the roots of the cubic equation (3.5a), from Theorem 1 we conclude that the structure parameters are related as in (D17). •

As can be seen from the expressions for the two roots in relation (D18), the solution for θ becomes unique when $V_x Z_Y = V_y Z_X$, i.e. the translation vector (V_x, V_y, V_z), the surface normal $(Z_X, Z_Y, -1)$ and the optical axis (the Z-axis) all lie in a plane.

D2.4 The four solution theorem

Theorem 7 : Under conditions of Theorem 2 if the cases (D6a,b) are false and the two solutions of equation (D12) for θ are also the roots of the cubic equation (3.5a) then up to four solutions are possible for the image flow equations (2.11a-l).

Proof : Here the common solutions of the equations (3.5a) and (D12) are the valid solutions for θ. Using the trigonometric identity

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{D19}$$

we can show that the solutions of equation (D12) are

$$\tan\theta = \frac{-a_2 \pm \sqrt{a_1^2 + a_2^2}}{a_1} \ . \tag{D20}$$

Expressing the terms on the right hand side of equation (D20) in terms of the structure and motion parameters (using relations (2.11c-f) and (3.2a,b)), the two roots for θ in this case can be shown to be

$$\tan\theta = \frac{V_y}{V_x} \ , \ -\frac{V_x}{V_y} \quad \left(OR \ \frac{Z_Y}{Z_X} \ , \ -\frac{Z_X}{Z_Y} \right) . \tag{D21}$$

Note that the two roots are such that their absolute difference is equal to $\pi/2$ radians. For each of the above two roots we may solve for r from either (3.5b) or (3.5c). In this case equations (3.5b) and (3.5c) are identical and can be written as

$$r^2 + \frac{1}{2}\left(\frac{v_{xx}}{s} + \frac{u_{yy}}{c} \right) r - V_z \frac{a_1}{2cs} = 0 \ . \tag{D22}$$

Equation (D22) gives up to two solutions for r for each θ. Therefore up to four solutions are possible in this case. ●

Lemma 3a : Under the conditions stated in Theorem 7 the structure parameters satisfy (D6c and) the following constraint:

$$-\frac{Z_X}{Z_Y} = \frac{1}{Z_{yy}} \left(-Z_{xy} \pm \sqrt{Z_{xy}^2 - Z_{xx} Z_{yy}} \right) . \tag{D23}$$

Proof : Since the two roots for θ given by (D21) are also the roots of the cubic equation (3.5a), from Theorem 1 we see that relation (D23) should be true of the structure parameters. ●

The image flow equations always have one solution which corresponds to the actual shape and motion of the surface in the physical world. Therefore when there are multiple solutions it is natural to try to relate the spurious solutions to the actual shape and motion of the surface. In order to do this we will adopt the following notation: we will denote the parameters corresponding to the actual or the ''correct'' solution by appending a ''0'' to the subscripts of the respective parameters, and we will denote the spurious solutions by appending distinct integers to these subscripts. For example, r_0 denotes the actual translation parallel to the image plane whereas r_1, r_2, ..., etc. denote the spurious solutions for the translation parallel to the image plane. Also, for the case considered in Theorem 7 and other cases (considered later) where the translation vector, the surface normal, and the optical axis all lie in a plane, we will define a quantity k such that

$$Z_X = kc \quad \text{and} \quad Z_Y = ks \ . \tag{D24}$$

Lemma 3b : Under the conditions stated in Theorem 7, the spurious solutions are related to the actual shape and motion parameters by

$$\theta_1 = \theta_0 \pm \pi/2 \ , \quad r_1 = -V_{z0} k_0 \ , \tag{D25a,b}$$

and r_2 and r_3 are the roots of the quadratic equation

$$r^2 + \frac{1}{2}\left(\frac{s_0 Z_{xx0}}{\pm c_0} + \frac{c_0 Z_{yy0}}{\mp s_0} \right) r + (V_{z0} r_0 k_0) = 0 \ . \tag{D25c}$$

Proof : Relation (D25a) is easily derived from the two solutions for θ given by (D21) (note that $\tan\theta_0 = V_y/V_x$). In order to derive (D25b), in equation (D22) we substitute $s \leftarrow s_0$ and $c \leftarrow c_0$, and substitute for all the image motion parameters in terms of r_0 and θ_0 (using relations (3.3a,b),(2.11i,j), (2.11e,f) and (D24)). Then using relation (D6c) we can derive

$$r^2+(V_{z0}k_0-r_0)r-(V_{z0}r_0k_0)=0 . \tag{D26}$$

The two roots of the above quadratic equation are r_0 and $-V_{z0}k_0$. Hence the relation (D25b). To derive relation (D25c) we first substitute $s \leftarrow s_1$, $c \leftarrow c_1$ in relation (D22). Then, as before, we substitute for all the image motion parameters and θ_1 in terms of r_0 and θ_0 to get relation (D25c). ●

Lemma 3c : Under the conditions stated in Theorem 7, for $\theta=\theta_0$ the solution for r is unique when the direction of translation is parallel to the surface normal.

Proof : In this case the solutions for r are the roots of the quadratic equation (D26) which are r_0 and $-V_{z0}k_0$. Using relations (3.3a,b) and (D24) we can easily show that these two roots are equal when the translation vector (V_x, V_y, V_z) is parallel to the surface normal vector $(Z_X, Z_Y, -1)$. ●

Lemma 3d : Under the conditions stated in Theorem 7, the spurious solution for θ given by relation (D25a) is a valid solution if and only if the roots of the quadratic equation (D25c) are real. ●

Using relations (D6c), (D23) and the condition that the roots of the quadratic equation (D25c) be real we have constructed an algorithm to generate numerical examples which result in up to four solutions for the image flow equations. One such example is given in Section 5 (Example 3).

D3. Multiple interpretations due to multiple solutions for *r*

In Section D2.4 we have seen one case where multiple interpretations arise because of multiple solutions for both θ and r. Next we consider another case of multiple interpretations characterized by a unique solution for θ but two solutions for r.

D3.1 Double solution theorem 2

Theorem 8 : Suppose that the curved-surface-condition is true and (D5a,b) are false. Then a solution for θ (obtained by solving equation (3.5a)) gives two solutions for *r* (and consequently for the image flow equations (2.11a-l)) *if and only if* the following conditions are true of θ :

$$\tan\theta = \frac{v_{yy} - 2v_0 + v_{xx}}{u_{xx} - 2u_0 + u_{yy}} \quad \text{and} \quad \tan 2\theta = \frac{a_1}{a_2} \tag{D27a,b}$$

Proof : We will first prove the *only if* part. For a given θ we solve for *r* by solving the two quadratic equations (3.5b,c) (each of which may yield up to two roots) and take the root(s) common to both as the solution. If there is no common root or the roots are complex then the given θ is not a valid solution. If there is one common root then the solution for *r* is unique. If both roots are (real and) common then the coefficients of the two quadratic equations have to be proportional (i.e. the two quadratic equations become linearly dependent). Equating the ratios of the corresponding coefficients of the two equations (3.5b,c) we have

$$\frac{2cs}{2cs} = \frac{v_{xx}c - (u_{xx} - 2u_0)s}{u_{yy}s - (v_{yy} - 2v_0)c} = \frac{s(a_1 s + a_2 c)}{c(a_1 c - a_2 s)} = 1\ . \tag{D28}$$

Therefore, equating the numerator and the denominator of the second term in equation (D28) we get condition (D27a) and equating the numerator and the denominator of the third term we get condition (D27b). The *if* part can be similarly proved. ●

Note that the equality of the numerator and the denominator of the third term in equation (D28) together with (3.3a,b) and (3.3e,f) implies

$$\tan\theta \;=\; V_y / V_x \;=\; Z_Y / Z_X\ . \tag{D29}$$

Equation (D29) implies that the translation vector $(V_x,\, V_y,\, V_z)$, the surface normal $(Z_X,\, Z_Y,\, -1)$ and the optical axis (the Z-axis) all lie in a common plane. Also note that we can check that there is no θ for which there are two solutions by first solving for θ from equation (D27a) and checking for the validity of (D27b) (or, the right hand sides of equations (D27a) and (D27b) can be directly related by using the relation (D19) between tanθ and tan2θ).

In this case, the condition for the solution for r to be unique can be derived as follows: since equations (3.5b) and (3.5c) are linearly dependent, we add them to get the following equation:

$$r^2+\frac{1}{4}\left(\frac{v_{xx}-v_{yy}+2v_0}{s}+\frac{u_{yy}-u_{xx}+2u_0}{c}\right)r-V_z\frac{a_1}{2cs}=0\ . \qquad \text{(D30)}$$

In terms of the actual structure and motion parameters, the above equation can be shown to reduce to equation (D26) whose roots are r_0 and $-V_{z0}k_0$. When these two roots are equal we can show that the translation vector is parallel to the surface normal (see Lemma 3c).

D4. Condition for uniqueness of interpretation

Theorem 9 : Suppose that the curved-surface-condition is true, (D5a,b) are false, and there is no θ which satisfies (D27a,b) (i.e. the translation vector and the surface normal do not lie in a plane) then there exists a unique solution for the image flow equations (2.11a-l).

Proof : To prove this, we simply observe that there is always one solution for θ computed by solving equation (3.5a) which corresponds to $\tan\theta = V_y/V_x$ and this θ and the corresponding r computed from, say, (3.5b) always satisfies the constraint equation (3.5c). Multiple solutions are ruled out due to the previous theorems. ●

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Index Terms

Three-dimensional interpretation of image flow, Optical flow.

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Figure captions

Figure 1. Camera model and coordinate systems.