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On Some Properties of One-Dimensional Resistive Spatial Filters

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Abstract — **The spatial stability and frequency response of one-dimensional resistive networks is studied. Both infinite and finite resistive grids (with various practical boundary conditions) are considered. The analysis is restricted to first- and second-order networks for which the spatial stability regions are determined. Conditions for achieving maximally-flat low-pass spatial filters are also derived.**

1 INTRODUCTION

Translation-invariant resistive networks like the one depicted in Fig. 1 have been used in various applications: image signal processing [1]-[3], analog computers for solving the discrete forms of Laplace and Poisson equations for different boundary conditions [4], and more recently, offset averaging in flash and folding analog-to-digital converters [5]-[11]. Some of the applications using resistive networks may require 2D grids (as in image processing or analog computing) and/or higher-order ones (resistors spanning more than two nodes). In this communication only 1D and first- and second-order resistive networks are considered. For the purpose of authors' intended final use of such grids (offset averaging in high-speed A/D converters) this will suffice.

High-speed A/D converters can be implemented in various architectures: flash, two-step (or multi-step), pipeline, folding/interpolating, and time-interleaved. Flash A/D converters do not require front-end sample-and-hold amplifiers and this makes them the best approach for high-speed applications. However, flash ADCs are limited to low resolutions because of power and silicon area constraints. The folding technique was introduced as a means to achieve the speed of the flash conversion at a fraction of the area and power consumption. It consists of an analog signal preprocessing that allows the reduction of the number of comparators in a flash-type converter. In both flash and folding ADC architectures the input signal bandwidth and sample rate can be increased if minimum size transistors are used in the input stages. Nevertheless, this will also increase the mismatch of the input devices and, consequently, the offsets of the amplifiers/comparators, therefore reducing the accuracy of the converters. An effective technique for decreasing the offsets in an array of amplifiers is resistive averaging that was initially proposed by Kattman and Barrow. [5]. Thus, resistors are connected between the outputs of amplifiers (the current sources in Fig. 1), acting as a filter that partially removes the "spatial noise" (the random offsets).

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2 THE INFINITE CASE

A detail of an infinite 1D second-order resistive spatial filter is shown in Fig. 1. The current sources $I[n]$, represent the inputs, whereas the node voltages $y[n]$ are the outputs. The resistor indexes show how many nodes the corresponding resistor spans. The network is described by the following difference equation:

$$\begin{aligned} y[n] &= a(y[n-1]+y[n+1])+b(y[n-2]+ \\ &\quad +y[n+2])+(1-2a-2b)x[n], \end{aligned} \tag{1}$$

where

$$a = \frac{G_1}{G_0+2G_1+2G_2}$$
$$b = \frac{G_2}{G_0+2G_1+2G_2}$$

and $x[n] = R_0 I[n]$ represent the node voltages when the lateral resistors are not present.

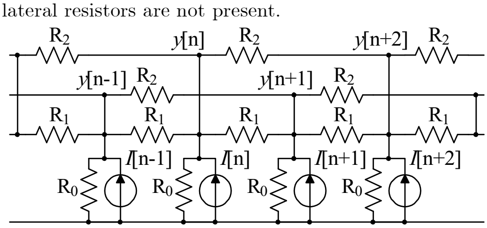


Figure 1. Detail of a 1D second-order resistive network.

(i) First-order network. When resistor R_2 is not present ($R_2 = \infty$), (1) describes a first-order non-causal IIR filter with the transfer function

$$H(z) = \frac{1-2a}{1-a\left(z+z^{-1}\right)} \tag{2}$$

and with the poles given by

$$z_{1,2} = \frac{1 \pm \sqrt{1-4a^2}}{2a}$$

Such a system is stable for $|a| < 0.5$ (for non-causal systems the instability is given by the presence of poles on the unit circle). The frequency response of the filter (2) is

$$H(\omega) = \frac{1-2a}{1-2a\cos\omega}$$

and for $0 < a < 0.5$ one obtains a low-pass filter. The bandwidth is set by the parameter a, the expression of the -3dB angular frequency being given by

$$\cos(\omega_{-3dB}) = 1 + \frac{\left(\sqrt{2}-1\right)(2a-1)}{2a}$$

The closer is a to 0.5, the narrower the filter.

A first-order resistive network with nearest-neighbor connections produce only an exponential convolution (the poles of the transfer function are real), equivalent to a poor selectivity in the frequency domain. Hence, extension to higher orders may be needed in practice.

(ii) Second-order network. In this case, the corresponding transfer function in Z transform is

$$H(z) = \frac{1-2a-2b}{1-a\left(z+z^{-1}\right)-b\left(z^2+z^{-2}\right)} \quad (3)$$

As it will be shown in the next sub-section, such a filter can have a higher roll-off slope in the stop-band and therefore can implement a more efficient spatial low-pass filter.

2.1 Spatial stability of infinite second-order resistive network

The stability of non-causal discrete-time systems is given by the absence of poles on the unit circle. In the following we will derive the acceptable values of parameters a and b that ensure the spatial stability of the second-order infinite resistive network. The poles of the transfer function for the infinite resistive grid are the solutions of the fourth-order equation

$$bz^4 + az^3 - 1 + az + b = 0 \quad (4)$$

Because it is much easier to derive the conditions when the roots of a fourth-order polynomial are purely imaginary numbers than if their absolute values are equal to unity, one applies the bilinear transformation to the above equation

$$z = \frac{s+1}{s-1}$$

This transformation maps the unit circle in the z plane onto the imaginary axes in the s plane. Therefore, determining if equation (4) has a solution on the unit circle is equivalent to check if the biquadratic equation

$$(2b+2a-1)\,s^4 + (12b+2)\,s^2 + (2b-2a-1) = 0$$

has imaginary solutions, which at its turn reduces to check if the second-order polynomial

$$P(u) = (2b+2a-1)\,u^2 + (12b+2)\,u + (2b-2a-1)$$

has real negative roots. Therefore, the values of the parameters (a, b) guaranteeing the stability of the non-causal spatial filter described by (3) have to satisfy the following set of inequalities

$$8b^2 + 4b + a^2 < 0$$
or
$$(2b+2a-1<0)\ \&\ (2b-2a-1<0)\ \&\ (6b+1>0)$$

and the region of stability corresponds to the interior of the contour depicted in Fig. 2. Because real poles determine an exponential convolution kernel, which corresponds to a poor selectivity in the frequency domain, efficient second-order filters will have parameters (a, b) inside the dotted line ellipse in Fig. 2. In this case parameter b is always negative, corresponding to negative resistors R_2 present in the network of Fig. 1. Nevertheless, efficient implementation solutions have been reported in literature to overcome this drawback [1], [11].

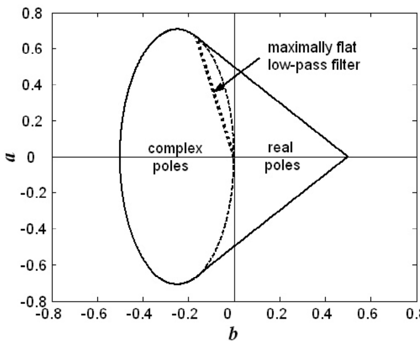


Figure 2. Spatial stability region.

2.2 Frequency response of the infinite second-order resistive network

The frequency response of the filter (3) is

$$H(\omega) = \frac{1-2a-2b}{1-2a\cos\omega-2b\cos 2\omega}.$$

To obtain a maximally flat low-pass filter, the derivatives of $H(\omega)$ at $\omega = 0$ should be zero for as many orders as possible, and the first non-zero one should be negative (to ensure the frequency response is a convex function). It can be easily seen that the first derivative $\partial H(\omega)/\partial\omega = 0$ for $\omega = 0$. The second derivative is

$$\left.\frac{\partial^2 H(\omega)}{\partial\omega^2}\right|_{/\omega=0} = \frac{2(a+4b)}{1-2a-2b}$$

and it nulls if $b = -a/4$. With the value of parameter b set this way, the third order derivative is zero for $\omega = 0$ and the fourth order one is

$$\left.\frac{\partial^4 H(\omega)}{\partial\omega^4}\right|_{/\omega=0} = \frac{12a}{3a-2}$$

which is negative if $0 < a < 2/3$. The -3dB bandwidth of the maximally flat second-order spatial filter is given by

$$\cos(\omega_{-3dB}) = 1 + \frac{\sqrt{\left(2-\sqrt{2}\right)a\,(2-3a)}}{2a}$$

The closer the parameter a to 2/3, the narrower the filter is. Combining these results with the ones regarding the stability of the network, in the design of the spatial filter the values of the parameters (a, b) will be chosen to lie on the straight line $4b + a = 0$ and inside the stable region (see Fig. 2). Figure 3 shows a comparative plot of the spatial frequency responses of the first- and second-order resistive networks. Both spatial filters have been designed to achieve the same -3dB bandwidth $\omega_{-3dB} = \pi/8$ and it can easily seen that the second-order network achieves a much steeper roll-off characteristic in the stop band.

Remark: The condition $b = -a/4$ has been also found in [2] by gradient descent optimization as a design requirement for obtaining spatial filters with Gaussian-like convolution kernel.

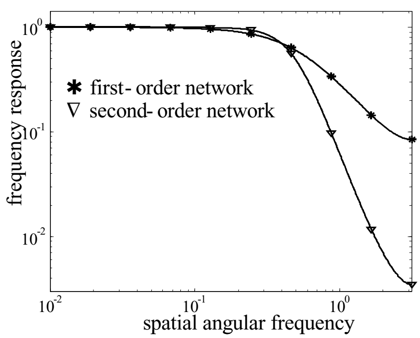
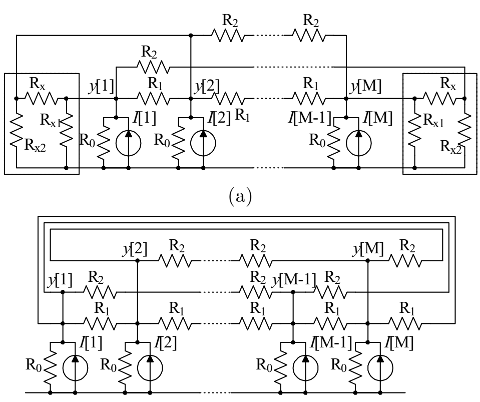


Figure 3. Spatial frequency response



(a)

(b)

Figure 4. Finite resistive networks.

3 THE FINITE CASE

The infinite network can be also described by the matrix equation

$$\mathbf{Ay} = (1 - 2a - 2b)\,\mathbf{x} \quad (5)$$

where $\mathbf{y}$ and $\mathbf{x}$ are column vectors containing the node voltages $y[n]$ and respectively $x[n] = R_0 I\,[n]$, and $\mathbf{A}$ is an infinite Toeplitz matrix with 1s on the main diagonal, $-a$ on the first subdiagonals, and $-b$ on the second subdiagonals. The node voltages $y\,[n]$ can be obtained as

$$\mathbf{y} = (1 - 2a - 2b)\,\mathbf{A}^{-1}\mathbf{x},$$

the columns of matrix $\mathbf{A}^{-1}$ containing the samples of the bilateral impulse response $h[n]$.

In a practical case, the resistive network is finite. There are two ways to obtain a finite resistive network from an infinite one:

(i) resistively terminating the grid with R_x, R_{x1} and R_{x2} (Fig. 4(a));

(ii) connecting the network in a ring (Fig. 4(b)).

Mathematically, this is equivalent to imposing some boundary conditions to (1).

3.1 Resistively terminated network

When terminating the network with grounded resistors, the top left corner of the matrix $\mathbf{A}$ (and, symmetrically, its bottom right corner) becomes

$$\begin{bmatrix} 1 - a - b + a_x + c_x & -a - c_x \\ -a - c_x & 1 - b + b_x \end{bmatrix} \quad (6)$$

where a_x, b_x, and c_x are given by

$$a_x = \frac{G_{x1}}{G_0 + 2G_1 + 2G_2}$$
$$b_x = \frac{G_{x2}}{G_0 + 2G_1 + 2G_2}$$
$$c_x = \frac{G_x}{G_0 + 2G_1 + 2G_2}$$

It is obvious that the newly obtained matrix is not Toeplitz anymore, but in a single case: $a_x = a$, $b_x = b$, $c_x = 0$ ($R_{x1} = R_1$, $R_{x2} = R_2$, and $R_x = \infty$), which corresponds to a particular case of terminating the network by imposing the boundary conditions $y[-1] = y[0] = y[M+1] = y[M+2] = 0$. Another possible solution is to leave in open circuit the connecting resistors at the edges of the finite network $R_x = R_{x1} = R_{x2} = \infty$ and this corresponds to $a_x = b_x = c_x = 0$. Both these terminations do not preserve the shift invariant property of the infinite network. In order to obtain a shift invariant finite network, one needs to optimally terminate the resistive grid by appropriately choosing the values of the resistors R_x, R_{x1}, and R_{x2}. The procedure for computing the optimal values of the terminating resistors is described in [12].

Such finite resistive networks can also exhibit spatial instability, but there is no closed analytical solution for the values of parameters (a, b) guaranteeing stability. Numerical simulations have shown that a resistively terminated finite network (at least in the cases of the three particular terminations mentioned above) obtained from a stable infinite network is also spatially stable.

3.2 Ring connected network

In the case of a ring connection (Fig. 4(b)), the matrix $\mathbf{A}$ in (5) becomes circulant. In this case, the resistance seen at all nodes is the same, therefore the network is automatically shift invariant. The response of the ring connected network is obtained by circular convolution from the input samples and the spatial impulse response of the infinite network. Therefore, a M-length ring connected network is spatially unstable if the poles of the transfer function (1) are given by $e^{j\omega}$, $\omega = \dfrac{2k\pi}{M}$, $k = 0, 1, 2, ...$, that is

$$a = \frac{1 - 2b\cos\left(\frac{4k\pi}{M}\right)}{2\cos\left(\frac{2k\pi}{M}\right)} \tag{7}$$

The values (a, b) satisfying (7) are located on lines spanning the instability region for the infinite network. The case corresponding to $M = 18$ is depicted in Fig. 5.

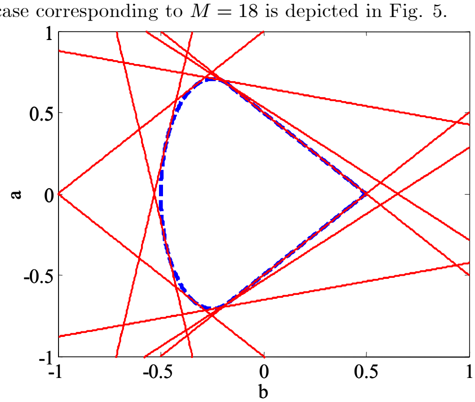


Figure 5. Stability region for a ring-connected network.

4 CONCLUSIONS

First and second-order infinite and finite resistive networks that implement non-causal spatial filters have been analyzed. It has been shown that maximally-flat low-pass spatial filters with increased roll-off characteristic can be implemented with active (negative resistors required) second-order resistive networks. In the case of infinite grids, the spatial stability is guaranteed by parameter values that are restricted to a closed convex set in the parameter space. For finite grids, independent of the approach of obtaining them, instability occurs for parameter values that lie on curves spanning the instability region of the infinite grids. Although stable finite resistive networks can be obtained from unstable infinite ones, these grids can exhibit temporal instability when parasitic capacitors are present between network nodes [12].

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