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Quantum percolation and quantization of Hall resistance in two-dimensional electron gas

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A theory of the quantized Hall resistance R_{xy} in a two-dimensional electron gas is presented. The exactness of the quantization is explained as a purely topological effect. It is shown that lines of constant electrostatic potential represent an effective wave guide for electron waves. In the Corbino ring geometry the condition for quantization of R_{xy} is the existence of equipotentials encircling the central electrode. The quantum of $R_{xy}=h/e^2$ is shown to be unaffected by a random scattering potential. Collapse of the Hall current on increasing disorder is interpreted as a percolation transition.

Recent experiments[1,2] with Hall resistance in semiconductor inversion layers in strong transverse magnetic fields established to the accuracy of better than 10^{-5} that Hall conductivity is quantized in multiples of e^2/h. The exactness of this result has led to the proposal[1] of using the quantum h/e^2 as a standard of resistance. The most fundamental consequence of these experiments is the existence of a long-range order in a fermionic system[3] which consists here in a self-interference of electrons over macroscopic distances of the sample. A good analogy would be to the Bohm-Aharonov experiment,[4] or to the double-slit interference of electrons propagating in vacuum a macroscopic distance from the slits to a screen. Laughlin[3] elegantly related the exactness of Hall quantization to gauge symmetry by considering the response of electrons on a two-dimensional metallic ribbon to varying magnetic flux through the ribbon. Still, his argument contains an intuitive assumption that the only consequence of adding an integral number of flux quanta hc/e is repopulation of current-carrying states. Although this is undoubtedly true for an ideal system it is not so obvious in the presence of disorder. Consider Laughlin's argument for a disordered system, say, with holes punched in the ribbon. Continuous variation of the vector potential leads to a mixing between the local and current-carrying states without which a state cannot be continuously dragged through a nonuniformity. Special consideration must be given to this situation because gauge symmetry by itself does not prevent the charge exchange between the local and the extended subsystems. The number of local states in quantum Hall experiments is by no means small: As follows from the data presented in Ref. 1 localized electrons constitute about 3% of the inversion layer charge (10—20% for Ref. 2).

In this paper we develop a physical picture of the Hall current in the presense of a random potential using a simple model which adequately describes localized states for the experimental situations of Refs. 1 and 2. In our picture the density $\vec{J}$ of the Hall current is proportional to the local electric field perpendicular to $\vec{J}$ in the plane of the inversion layer. Exactness of the quantization is a topological property of equipotentials in this plane. We describe the quantum Hall current as a surface effect in which edge states play no important role. This approach differs from that taken in Ref. 5 where the Hall current was interpreted as a net difference between two oppositely directed edge currents which were assumed to flow (exactly canceling each other) even in the absence of an applied voltage. It is an open question whether or not edge currents exist in a particular experimental arrangement. We would like to point out, however, that when they do exist they should not only contribute to the Hall current but also give rise to a novel type of diamagnetism, experimentally distinguishable from the conventional Landau effect.[6]

Consider a two-dimensional electron gas (2DEG) of the Corbino ring geometry in a strong transverse magnetic field H. First, let us discuss an ideal situation with no lateral fluctuation of the electrostatic potential. In the absence of an applied electric field the electron energy is characterized by quantum numbers i and n, viz.,

$$E_{in}=(n+\tfrac{1}{2})\hbar\omega_c+E_i\ , \tag{1}$$

where the cyclotron frequency $\omega_c=eH/m^*c$. Equation (1) differs from the usual expression[7] for

a three-dimensional electron gas in that the continuous energy spectrum $p_z^2/2m^*$ characterizing the free motion in the direction of H is replaced by discrete energies E_i appropriate for a finite motion in the quantum well confining electrons in the inversion layer. The level splitting in this quantum well is assumed to be so large ($\gg\hbar\omega_c$) that all electrons remain frozen in states with $i=0$ for all H considered. For the sake of clarity we shall neglect spin effects. The levels E_{in} are degenerate with the number of states in each level given by $N=eHS/hc$ where S is the area of the 2D EG sample. Thus each filled level contributes $\sigma=e^2H/hc$ to the surface charge density in the inversion layer ($\sigma/e=1.3\times10^{11}$ cm^{-2} for $H=5$ T). We shall consider only the case when $\hbar\omega_c\gg kT$ so that all states below the Fermi level E_F are completely filled and those above E_F empty. States belonging to the same degenerate level can be labeled by an additional quantum number x_0 which corresponds to one of the coordinates of a classical cyclotron orbit. Along the x axis the electron wave function varies like the nth eigenfunction of a linear oscillator centered at x_0, and is localized with a dispersion

$$a_n^2\equiv\langle n\,|\,(x-x_0)^2\,|\,n\rangle=(n+\tfrac{1}{2})a_L^2\ ,$$

where the Landau length $a_L\equiv(\hbar c/eH)^{1/2}$ (for $H=5$ T, $a_L\approx100$ Å). In the perpendicular direction the wave function is $\exp(ip_yy)$ where $p_y=eHx_0/c$. These results of Landau are strictly valid in a Cartesian system while our coordinates are only locally Cartesian which produces a negligible error of order a_L^2/R^2 (where R is the radius of the Corbino ring) compared to the exact solution in the cylindrical coordinate system (cf. Ref. 7). The choice of direction in which orbits are localized is determined by the gauge for the vector potential $\vec{A}$ of the magnetic field and is quite arbitrary.

When there is a radial electric field F, i.e., when a voltage V_0 is applied, then it is convenient to use a gauge in which $A=A_y=Hx$, with the local x axis directed along $\vec{F}$. It is easily seen than in this case the Schroedinger equation is satisfied by wave functions of the form similar to those obtained in the absence of the electric field but with a different relation between x_0 and the momentum p_y of the electron wave, viz.,

$$p_y=(eH/c)x_0+m^*cF/H \tag{2}$$

and a different energy spectrum,

$$E_{inx_0}=E_{in}+eFx_0+\tfrac{1}{2}m^*(cF/H)^2\ . \tag{3}$$

As seen from Eq. (3) the Landau levels are split by electric field. Each quantum number x_0 determines an equipotential. The electronic waves are localized in z direction by the quantum well and in x direction by the length a_n, while in the y direction they propagate along the equipotential lines like light in an optical fiber. Each closed equipotential represents a ring resonator which imposes a cyclic boundary condition on the electronic wave. This results in a discrete spectrum of values for x_0, with a step $\delta x_0=2\pi a_L^2/L$, where $L\gg a_L$ is the length of the fiber. Two successive orbits separated by the infinitesimal distance δx_0 also differ in the total variation of the phase of the corresponding wave functions on going around the loop. For the wave function to be single-valued this phase variation must equal $2\pi l$ (with l integer) for any state. The absolute value of l is gauge dependent but for two successive orbits the difference $\delta l=1$, which makes the two states orthogonal. It can be easily shown that the magnetic flux through the area bounded by two corresponding equipotentials equals hc/e. The first term in Eq. (2) gives no contribution to the total current due to a single electron, because it is exactly compensated by the diamagnetic term in the expression for the current density in a magnetic field.[7] Each electron contributes to the Hall current only in virtue of the additional velocity $v_H=cF/H$ it acquires in the electric field. This velocity also gives rise to the kinetic energy term in Eq. (3). Each filled Landau level contributes a linear current density $J=\sigma v_H$ and the total current in y direction is thus

$$I_0=\bar{n}(e^2/h)V_0\equiv G_{xy}V_0\ , \tag{4}$$

where $\bar{n}$ is the number of filled Landau levels and G_{xy} the transverse conductivity. We see that for a given V_0 the Hall current does not depend on the width of the ring. Although the number of quantized equipotentials (single electron fibers) is smaller for a narrower ring, the field F and therefore the Hall velocity v_H of each electron are proportionally increased so that the total current remains the same.

Even more remarkable is the fact that the Hall current is unaffected by a random electrostatic potential in the inversion layer. This is a purely topological effect as is demonstrated below. To be specific we consider the experimental situation of Ref. 9 in which the vanishing longitudinal resistance effect was first discovered. In this work the inversion layer was formed on the p-GaAs—

n-GaAlAs interface due to a work-function difference. Owing to random variations in the surface density of the fixed charge the shape and the depth of the potential well confining the inversion layer also fluctuate and the energies E_i become ill-defined. However, an important feature of this structure is an undoped GaAlAs layer of thickness d separating the fixed donor charge from the inversion layer which is formed on the lightly doped p-GaAs side of the heterojunction. If $d \gg a_L$ then the energies E_i become smooth functions of the lateral position in the inversion layer and the quantity E_0/e becomes equivalent to a random electrostatic potential for the 2D electrons. In this case the quantum mechanics of the problem can still be described in local Cartesian coordinates formed by the orthogonal grid of equipotential and field lines but the equipotentials are no longer concentric circles, cf. Fig. 1. Each equipotential represents an effective wave guide where an electron is localized to within the Landau length a_L. As before, periodic boundary conditions determine the quantization of equipotentials and turn the latter into fiber ring resonators. Topologically, there are two distinct classes of fibers in the geometry of a Corbino ring: *global* fibers which encircle the central electrode, and *local* fibers, which can be contracted to a point by a continuous deformation. It is the existence of global fibers which embodies the long-range order in the 2D EG. Because of the potential fluctuation the total number of electrons contributing to the Hall current is reduced, since local fibers obviously do not contribute. Moreover, in the region bounded by a local equipotential non-

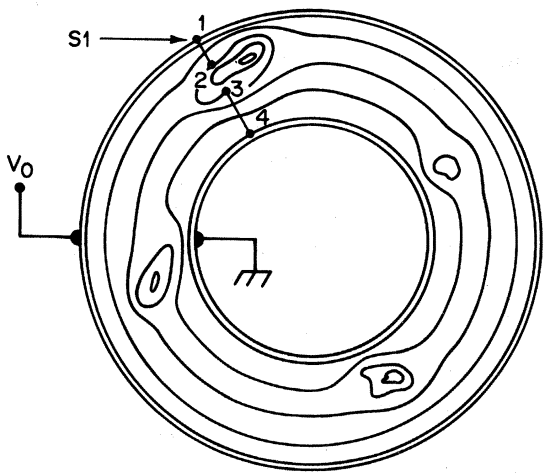

FIG. 1. Global and local fiber states in a Corbino sample.

fiber states can also exist but their presence does not affect the following argument.

In the presence of localized states the current remains the same as in the ideal situation. Indeed, consider a radial section of the sample which crosses one or more isolated closed loops, e.g., section $S1$ in Fig. 1. Because points 2 and 3 lie on an equipotential the sum of voltages dropping in regions $1\rightarrow 2$ and $3\rightarrow 4$ equals the applied voltage V_0. The effective width of the Corbino ring in section $S1$ is therefore reduced by the distance $2\rightarrow 3$. However, the Hall current for a given V_0 does not depend on the width of the ring, cf. Eq. (4). It may appear that the accuracy of this argument is influenced by the curvature of a fiber which limits the applicability of local Cartesian coordinates. Indeed, the state of the motion tranverse to the fiber (local x direction) is represented by a linear oscillator wave function only to the accuracy of a_L^2/r^2 where r is a local curvature radius. Nevertheless, as will be now rigorously proven, the accuracy of the Hall resistance quantization is far greater.

Consider a strip of thickness a_L along a global fiber of length L. To the accuracy a_L/L this strip can be regarded as a linear conductor, for which the current I_s and the associated flux Φ_s of magnetic field through the contour I_s are complementary thermodynamic variables. Therefore,

$$I_s = c\frac{\partial G_s}{\partial \Phi_s}\ , \tag{5}$$

where G_s is the free energy of electrons in the given strip. The single-electron contribution to the total current is given by

$$I_1 \equiv \delta I(x_0) = \frac{\partial I_s}{\partial N_s}\ ,$$

with N_s being the number of electrons in the strip. On the other hand, $\partial G_s/\partial N_s \equiv \mu(x_0)$ where μ is the chemical potential of electrons in the strip. This is a thermodynamic relation valid to within $1/N_s$ which is again a quantity of order a_L/L. Differentiating Eq. (5) we have $\delta I(x_0) = c\partial\mu/\partial\Phi_s$.

It should be emphasized that the flux of the magnetic field through any fixed area of the ring is *not* quantized and in contrast to the situation familiar in superconducitivy it can vary continuously. The fundamental difference is in the nature of the diamagnetism which in the present case is the Landau diamagnetism of the electron gas. The magnetic flux through a hole in a superconducting ring can vary only discontinuously because of its screening by a macroscopic diamagnetic current on

the inner surface of the ring. Such a coherent macroscopic current can exist in a superconductor only in virtue of the bosonic nature of the carriers (Cooper pairs) which can multiply occupy the same quantum state. The Landau diamagnetism, on the other hand, is not due to any macroscopic currents but to the spatial correlations of current densities. What is quantized in the present case is the magnetic flux through a variable area bounded by two global orbits on the chosen strip. The minimum flux variation $\delta\Phi_s$ corresponds to adding one extra electron to the strip and equals $\delta\Phi_s = hc/e$. This magnitude of the flux "quantum" follows from the periodic boundary conditions on the wave functions of current-carrying states (which implies that the flux increment must be hc/e times an integral number δl) and the principle of least action (whence $\delta l = 1$). The corresponding quantum of the chemical potential at zero temperature represents the variation of the Fermi energy on filling one successive quantized orbit, $\delta\mu = eF\delta x_0$. We thus find

$$\delta I(x_0) = (e/h)\delta\mu = (e^2/h)F\delta x_0 \ . \tag{6}$$

Summing over all filled global fibers, e.g., between points $1 \rightarrow 2$ and $3 \rightarrow 4$, we again arrive at Eq. (4). This proves the exactness of Hall quantization at least to the accuracy of a_L/L, with L being a macroscopic distance of the order of the length of the Corbino ring.

At a finite temperature T in addition to the Hall current $I_y = I_0$ there is a longitudinal current I_x due to generation of mobile carriers, i.e., thermal excitation of "electrons and holes" across the Landau gap $\hbar\omega_c$. With decreasing temperature this current goes to zero as $\exp(-\hbar\omega_c/kT)$ and so does all dissipation. The vanishing longitudinal conductivity G_{xx} implies that the longitudinal resistance

$$R_{xx} = G_{xx}/(G_{xx}^2 + G_{xy}^2)$$

also vanishes. However, this phenomenon is different in principle from superconductivity and not only in that the latter occurs discontinuously at a finite temperature. An important difference is in the nature of the long-range coherence which in the case of superconductivity consists in the rigidity of the phase of the Cooper-pair system wave function. In the present case, because of the energy splitting, the electron waves oscillate at different frequencies and one cannot speak of a common phase of oscillation. On the other hand, for a single fermion the phase is never a quantum number because of the Pauli principle and the uncertainty relation between the phase and the particle number.

Variable-range hopping between localized states (in our case, the local fibers) also contributes to a dissipative current along the electric field. Indeed, if the area bounded by a local fiber represents a potential hill or a potential "volcano" with its top above the Fermi level, then certain local fiber states will not be occupied. Some of these states with energies close to the Fermi level take part in the hopping current. Temperature dependence of this current is described by Mott's law $G_{xx} \propto \exp[-(T_0/T)^{1/3}]$ for a two-dimensional system.[9] At a sufficiently low temperature or high degree of disorder this path of current dominates over the generation current. Thus, the temperature dependence of R_{xx} obtained in Ref. 8 can be explained by the Mott conductivity between local fibers.

An interesting phenomenon will occur if we further increase the disorder. When the effective width of the Corbino ring goes to zero (becomes less than a_L) at least in one cross section then *all* global fibers are squeezed out and the macroscopic Hall current ceases. This phenomenon can be interpreted as a first-order phase transiton of percolation type with the Hall current playing the role of an order parameter. It should be noted, however, that no real percolation of particles is involved and the term "percolation" is used in a new, quantum, sense to describe the penetration of extended electronic orbits over the entire sample.

The above theory has one important limitation. We have considered the situation in which the only statistically important states are the global and the local fibers as well as any nonfiber states surrounded by a local equipotential. We recall that by fiber states we mean those associated with equipotentials smooth on the scale of a_L. All quantum Hall experiments known to us conform to this situation (Refs. 1, 2, and 8). However, it is conceivable that global equipotentials rugged on the scale of order a_L may exist and be statistically important in some experimental arrangements. It is not clear whether the quantized Hall effect will be observed in this case. In this respect of particular interest are the works of Ando[10] and Prange[11] which considered the electron scattering by short-range potential fluctuations.

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