

DIFFUSION IN A SHORT BASE

ANATOLY A. GRINBERG and SERGE LURYI

AT&T Bell Laboratories, Murray Hill, NJ 07974, U.S.A.

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Abstract—The usual diffusion-equation description of transport in the base of a junction transistor breaks down if the base length is comparable to the minority carrier mean free path. We present a rigorous analytic treatment of this problem, based on an exact solution of the Boltzmann transport equation (BTE). A key ingredient of our approach is formulation of the boundary conditions for the distribution function $f(\mathbf{r}, \mathbf{k}, t)$ at the base–emitter interface. Numerical solution of the BTE shows that there are significant corrections (of order 100%) to the diffusion-equation estimates of both the static current gain and the frequency cut-off in a short-base bipolar junction transistor (BJT). In a model where the electron scattering mean free path and the recombination length are both assumed independent of energy, the steady-state BTE reduces to an integral equation for the electron concentration. We present an approximate analytical solution of this equation that gives an excellent agreement with the exact numerical solution. The analytic solution is asymptotically exact in the limit of ultra-short base lengths, where the minority-carrier transport can be regarded as thermionic, as well as in the long-base limit (where the diffusion equation is rigorously valid). On the basis of our analytic solution, we propose and test a new expression for the effective (concentration-dependent) diffusivity, that interpolates between the diffusive and thermionic limits.

1. INTRODUCTION

Electrical characteristics of bipolar junction transistors (BJTs) are in large part determined by the transport of minority carriers across the base. For not too high injection currents, the base is quasi-neutral and the minority-carrier transport is described classically as a neutral-particle diffusion process. The concept of diffusion across a region of length W is valid only if the particles undergo multiple collisions during their characteristic transport time $\tau_{\text{diff}} = W^2/D_0$, where D_0 is the diffusion coefficient. Since $D_0 \sim v_T l_{\text{sc}}$, the condition $W \gg l_{\text{sc}}$ is equivalent to the requirement that the effective transport velocity W/τ_{diff} be much less than v_T . Here v_T is a thermal velocity of carriers, T the temperature, k the Boltzmann constant, m the effective mass of the carriers and l_{sc} is the scattering length. In the opposite limit, $W \lesssim l_{\text{sc}}$, the transport is largely ballistic. Inadequacy of the diffusion approximation in this limit manifests itself in regions of low concentration n and high concentration gradient, where the effective diffusion velocity $n^{-1}D_0 \nabla n$ may become larger than the thermal velocity.

It had been noted long ago [1–3] that the cut-off frequency f_T in high-frequency transistors is usually below theoretical predictions based on the diffusion approximation for the minority-carrier transport in the base. Having correctly observed that the problem arises from a breakdown in the validity of the diffusion equation in regions of high carrier concentration gradient, Persky [1] proposed an empirical formula for “diffusion saturation”, which smoothly replaces the diffusion flux $D_0 dn/dz$ by a thermionic flux nv_R

when the former becomes comparable to or exceeds the latter. Here $v_R = (kT/2\pi m)^{1/2}$ is a particular form of the thermal velocity (often referred to as the Richardson velocity), corresponding to the mean velocity of carriers moving in the positive z -direction in equilibrium. Persky’s formula is supposed to interpolate between the regimes of diffusive and thermionic transport (see, however, the discussion in Section 5 below). A somewhat different formula was proposed by Berz [3] who derived it from a microscopic model, assuming a Maxwellian distribution of carrier velocities. It may be worthwhile to note that Berz’s model corresponds to the so-called Milne’s problem (transport of neutral particles subject to elastic collisions in a semi-infinite medium with a perfect sink at the boundary) and it admits of an exact solution, well-known in the neutron transport theory.

In the case of a junction transistor, the diffusion saturation effects can be expected within a region of a few mean free paths l_{sc} from the base–collector junction, which acts as a sink for minority carriers. The problem becomes important, therefore, when this region becomes comparable to the total base thickness W , while in the limit $W \gg l_{\text{sc}}$ the usual description on the basis of the macroscopic diffusion equation is accurate. In order to properly account for the effects arising from a breakdown of the diffusion equation, it is essential to consider the transport across a thin slab (rather than semi-infinite medium). A natural framework for this would be to start with an exact solution of the Boltzmann transport equation in the base slab. Such a consideration, however, requires a reasonable approximation for the

boundary conditions on $f(\mathbf{r}, \mathbf{k}, t)$ at the emitter–base and the base–collector junctions. This has proven to be a tricky problem, since the boundary functions themselves depend on the solution in the base. For example, in a stationary problem, the full functions $f(0, \mathbf{k})$ and $f(W, \mathbf{k})$ uniquely determine the emitter and the collector current, respectively, and hence if we had known these functions, no further work would be necessary for many practical problems. The progress in analytic modeling of the Boltzmann transport in the transistor base seems to have been impeded by the lack of convenient boundary conditions on the distribution function. This problem does not arise in Monte-Carlo treatments[4,5] that can afford to impose the boundary conditions deep inside the emitter and collector layers, where the distributions are near equilibrium to a good approximation.

In the present work, we propose a simple way around this problem. The boundary conditions we derive represent a natural generalization of those underlying Bethe's thermionic theory of barrier transport. Namely, we assume that at some distance into the emitter–base junction within less than a mean free path from the "beginning" of the proper region, the electron distribution is in an approximate equilibrium with the emitter. In other words, we neglect the variation of the electron imref $E_{F,n}$ in the region $z < -z_0$ where $z_0 < l_{sc}$, cf. Fig. 1. Conditions for the validity of this assumption are similar to those for the validity of the thermionic theory in Schottky diodes, namely that the electric field $F(z_0)$ is sufficiently large, but a subtle conceptual change should be made, cf. Section 2, in the usual formulation of these conditions[6].

The assumption of equilibrium at z_0 translates into a definite boundary condition $f(0, \mathbf{k}; \mathbf{k} \cdot \mathbf{z} > 0)$ that describes only a *part* of the electronic ensemble at $z = 0$ with velocities directed into the base. On the base–collector boundary we assume a perfect-sink condition, $f(W, \mathbf{k}; \mathbf{k} \cdot \mathbf{z} < 0) = 0$, that applies to the

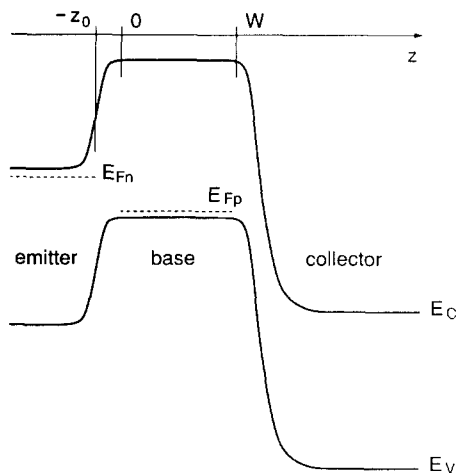


Fig. 1. Schematic band diagram.

complementary part of the electronic ensemble. Neither of these two functions determines the carrier concentration, the drift velocity or the current at the respective boundaries, but combined they constitute an adequate set of boundary conditions for the Boltzmann transport equation in the base.

This paper is organized as follows. In the next section the boundary conditions are mathematically formulated. We shall be treating a 1-D problem, assuming homogeneity in the x - y plane. Section 3 introduces a particular model for the collision integral which assumes energy-independent lengths of electron scattering and capture. The former assumption is strictly justified only for collisions of carriers with acoustic phonons. However, we believe that the qualitative picture following from our analysis is rather general and it depends only weakly on the concrete scattering mechanism. In the present model, the Boltzmann transport equation *including the boundary conditions* is brought into the form of an integral equation for the spherically symmetric part $f_0(z, E_{\mathbf{k}}, t)$ of the distribution function. Section 4 describes the solution of the steady-state problem. The results are compared with those following from the diffusion equation. Section 5 discusses possible generalizations of the diffusion equation, introducing an effective diffusivity that depends on the local concentration and concentration gradient. Section 6 deals with the small-signal frequency response; the frequency cut-off f_T of the transistor current gain is calculated for a wide range of base thicknesses. Our conclusions will be summarized in Section 7.

2. BOUNDARY CONDITIONS FOR THE MINORITY-CARRIER DISTRIBUTION FUNCTION IN THE BASE

Referring to Fig. 1, let us assume that within a scattering mean free path l_{sc} from the base origin at $z = 0$ there is a plane $z = -z_0$ where the electron distribution function can be taken in the Maxwellian form, corresponding to an approximate equilibrium with the electronic ensemble in the emitter. Conditions for the validity of this assumption will be discussed below. In the region $-z_0 < z < 0$ the steady-state distribution function satisfies:

$$\frac{\hbar k_z}{m} \frac{\partial f}{\partial z} + \frac{eF_z}{\hbar} \frac{\partial f}{\partial k_z} = 0. \quad (1)$$

Solution of this equation is an arbitrary function of the form:

$$f = \phi(k_x, k_y, [2meV(z)/\hbar^2 + k_z^2]^{1/2}) \quad \mathbf{F} \equiv -\partial V/\partial \mathbf{r}. \quad (2)$$

Therefore, if we know that $f(-z_0, \mathbf{k})$ is an equilibrium Maxwellian function, we can conclude that for $\mathbf{k} \cdot \mathbf{z} > 0$ the function $f(0, \mathbf{k})$ is of the form:

$$f(0, \mathbf{k}) = \frac{n^*}{N_C} \exp\left[-\frac{E_{\mathbf{k}}}{kT}\right], \quad \mathbf{k} \cdot \mathbf{z} > 0, \quad (3)$$

where $N_C = 1/4(2mkT/\pi\hbar^2)^{3/2}$ is the conduction band density of states, $E_k \equiv \hbar^2 k^2/2m$:

$$n^* = n_E \exp[-e(\Phi - V_{BE})/kT], \quad (4)$$

n_E is the majority-carrier concentration in the emitter, V_{BE} is the applied base-emitter voltage and $\Phi > 0$ is the built-in potential in the base-emitter junction.

Even though eqn (4) formally coincides with a standard expression in the bipolar transistor theory, n^* does not represent the minority-carrier concentration $n(0)$ at the emitter side of the base. The actual carrier concentration includes the contribution of electrons with $\mathbf{k} \cdot \mathbf{z} < 0$, and if these were absent (as would be the case in Schottky contact neglecting reflections) then we would find from eqn (3) that $n(0) = n^*/2$. This conclusion is in excellent agreement with the results of Monte-Carlo simulations of Schottky diodes[7], according to which the velocity distribution at the top of the barrier is very close to a unidirectional Maxwellian distribution with a mean velocity towards the metal given by $2 \cdot v_R$ and the concentration $0.5n^*$, so that the resulting current density is still given by the standard Richardson formula. Similar results were found in Monte-Carlo simulations[8] of triangular (planar-doped) barriers; for those barriers, however, the reflected electrons make a non-negligible contribution and the simulated electron concentration on the top of the barrier lies between $0.5n^*$ and n^* .

Let us now discuss the validity of our assumption of an equilibrium distribution at the plane $z = -z_0$ in the emitter-base junction. A similar assumption underlies the thermionic theory of barrier injection[6]. As discussed above, the consequences usually drawn from this assumption—regarding the carrier concentration and the mean velocity on the top of the barrier—must be revised but the final result (Richardson formula) remains valid due to a cancellation of two factors of 2. The assumption itself is justified if the net flux of electrons at z_0 is much smaller than either of the oppositely directed diffusion and drift fluxes. That condition is ensured by the inequality $D_0(\partial n/\partial z) \gg n v_R$ which, in light of the Einstein relation, $eD_0 = \mu kT$, is equivalent to:

$$\mu F \gg v_R \equiv (kT/2\pi m)^{1/2}, \quad (5)$$

where μ is the equilibrium electron mobility. Another way of stating this condition is to require that the field $F(z_0)$ is sufficiently large for the potential gain over a mean free path to much exceed the thermal voltage:

$$Fl_{sc} \gg kT/e. \quad (6)$$

Equations (5) and (6) are mathematically equivalent, because in general one has $D_0 \sim v_T l_{sc}$ (Ref. [9], p. 40), $v_R \sim v_T$ and hence $\mu kT \sim e v_R l_{sc}$.

The boundary condition (3) should be complemented by another condition at the base-collector

interface for $\mathbf{k} \cdot \mathbf{z} < 0$. These conditions combined do not overspecify the problem of finding the distribution function $f(\mathbf{r}, \mathbf{k})$. It is clear that the distribution of minority carriers *incident* on the base region from outside, i.e. the values of $f(0, \mathbf{k})$ for $\mathbf{k} \cdot \mathbf{z} \geq 0$ and of $f(W, \mathbf{k})$ for $\mathbf{k} \cdot \mathbf{z} \leq 0$, gives a complete and non-contradictory input to a physical description of the transport process of interest to us. This represents a special case of the uniqueness theorem for solution of the transport equation in a bounded volume[10].

In this work we shall assume that on the base-collector boundary the function $f(\mathbf{r}, \mathbf{k})$ satisfies the "perfect sink" condition:

$$f(W, \mathbf{k}) = 0, \quad \mathbf{k} \cdot \mathbf{z} < 0. \quad (7)$$

It is easy to generalize this condition to the case of a partially reflecting boundary[3].

3. INTEGRAL FORM OF THE TRANSPORT EQUATION IN THE BASE, INCLUDING BOUNDARY CONDITIONS

The general form of the Boltzmann transport equation is:

$$\frac{\partial f(\mathbf{r}, \mathbf{k}, t)}{\partial t} = -\frac{1}{\hbar} \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{k}} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{e}{\hbar} \frac{\partial V}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{k}} + [\hat{S}_{\mathbf{k}}^{(+)} - \hat{S}_{\mathbf{k}}^{(-)}] f(\mathbf{r}, \mathbf{k}, t), \quad (8)$$

where $\hat{S}_{\mathbf{k}}^{(+)}$ and $\hat{S}_{\mathbf{k}}^{(-)}$ are the collision operators describing, respectively, the scattering into and from a point \mathbf{k} of the momentum space.

We shall assume that the system is homogeneous in the scattering parameters, that the electric field is zero, $\partial V/\partial \mathbf{r} = 0$ and that the distribution is inhomogeneous only in the z -direction. We shall also assume a periodic perturbation with an angular frequency ω ; since the equation is linear, we can look for a solution in the form:

$$f(\mathbf{r}, \mathbf{k}, t) = f(z, k, u) e^{i\omega t}, \quad (9)$$

where u is the cosine of the angle between k and the z -axis.

The outgoing collision integral $\hat{S}_{\mathbf{k}}^{(-)} f$ can be generally represented in the form:

$$\frac{m}{\hbar k} \hat{S}_{\mathbf{k}}^{(-)} f = \frac{f(\mathbf{r}, \mathbf{k}, t)}{l_{\text{tot}}(k)}. \quad (10)$$

This term can include not only scattering processes but electron captures as well:

$$\frac{1}{l_{\text{tot}}} = \frac{1}{l_{sc}(k)} + \frac{1}{l_{cp}(k)}, \quad (11)$$

where $l_{cp}(k)$ is the scattering length corresponding to the capture processes.

The incoming collision integral $\hat{S}_{\mathbf{k}}^{(+)} f \equiv K_i(z, \mathbf{k})$ is in general a complicated functional on f . For the scattering of electrons by phonons, it can be

expressed in terms of the symmetric part f_0 of the distribution function:

$$f_0(z, k) = \frac{1}{4\pi} \int f(z, \mathbf{k}) d\mathbf{k} = \frac{1}{2} \int_{-1}^1 f(z, k, u) du. \quad (12)$$

Interaction with optical phonons leads to an expression containing f_0 at energies $E_{\mathbf{k}} \pm \hbar\omega_{\text{op}}$. For the scattering by charged impurity centers, the function K_f is expressed by an integral over u of a weighted distribution $f(z, \mathbf{k})$. These microscopic expressions are adduced in Appendix A. Exact inclusion of either the optical phonon or the impurity scattering leads to complicated equations and will not be pursued in this work. We shall confine ourselves to the simplest case, when the incoming collision integral can be approximated by the following expression:

$$\frac{m}{\hbar k} \hat{S}_{\mathbf{k}}^{(+)} f = \frac{f_0(z, k)}{l_{\text{sc}}(k)}. \quad (13)$$

This expression is exact for scattering due to acoustic phonons only (in the quasi-elastic approximation), in which case, moreover, the scattering length is not a function of the electron energy, $l_{\text{sc}} \neq l_{\text{sc}}(k)$. Within a reasonable approximation (see Appendix A), the collision integral can also be reduced to the form (13) for scattering by charged impurity centers.

Substituting (9) into eqn (8), we obtain:

$$u \frac{df(z, k, u)}{dz} + \frac{f(z, k, u)}{l^*(k, \omega)} = \frac{f_0(z, k)}{l_{\text{sc}}(k)}, \quad (14)$$

where

$$\frac{1}{l^*(k, \omega)} \equiv \frac{i\omega m}{\hbar k} + \frac{1}{l_{\text{tot}}(k)}. \quad (15)$$

Equation (14) is a linear differential equation, describing the coordinate dependence of f for each value of the parameters ω , k and u . Therefore, the boundary conditions themselves may depend on these parameters. Using the boundary conditions (3) and (7), we find:

$$u > 0: f(z, k, u) = \frac{n^*}{N_C} \exp(-E_{\mathbf{k}}/kT) \times \exp(-z/ul^*) + \frac{1}{u} \int_0^z \frac{f_0(z', k)}{l_{\text{sc}}(k)} \times \exp[-(z-z')/ul^*] dz'; \quad (16a)$$

$$u < 0: f(z, k, u) = -\frac{1}{u} \int_z^{\infty} \frac{f_0(z', k)}{l_{\text{sc}}(k)} \times \exp[-(z-z')/ul^*] dz'. \quad (16b)$$

Substituting eqns (16) into (12), we obtain an integral equation for $f_0(z, k)$:

$$f_0(z, k) = \frac{n^* \exp(-E_{\mathbf{k}}/kT)}{2N_C} E_2 \left[\frac{z}{l^*(k, \omega)} \right] + \frac{1}{2} \int_0^{\infty} \frac{f_0(z', k)}{l_{\text{sc}}(k)} E_1 \left[\frac{|z-z'|}{l^*(k, \omega)} \right] dz', \quad (17)$$

where the functions $E_n(x)$ are defined by (see Appendix B):

$$E_n(x) = \int_0^1 t^{n-2} \exp(-x/t) dt. \quad (18)$$

It is convenient to introduce an energy density $G(z, k)$ of the electron current $J(z)$:

$$J(z) = \int_0^{\infty} G(z, k) dE_{\mathbf{k}}. \quad (19)$$

With the help of eqns (16) we find the following expression for $G(z, k)$:

$$G(z, k) \equiv \frac{-2e}{(2\pi)^3} \int \frac{\hbar k'}{m} f(z, k', u) u' \delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) d^3\mathbf{k}' \\ = \frac{-emE_{\mathbf{k}}}{\pi^2 \hbar^3} \int_{-1}^1 f(z, k, u) u du \\ = \frac{-emE_{\mathbf{k}}}{\pi^2 \hbar^3} \left[\frac{n^* \exp(-E_{\mathbf{k}}/kT)}{N_C} E_3 \left(\frac{z}{l^*(k, \omega)} \right) + \int_0^{\infty} \frac{f_0(z', k)}{l_{\text{sc}}(k)} E_2 \left(\frac{|z-z'|}{l^*(k, \omega)} \right) \times \text{sign}(z-z') dz' \right]. \quad (20)$$

4. STEADY-STATE SOLUTION

In a steady-state problem, $\omega = 0$ and $l^* = l_{\text{tot}}$. Using the properties of $E_n(\zeta)$, discussed in Appendix B, it can be shown that in the absence of capture processes ($l_{\text{cp}}^{-1} = 0$) the current, calculated from eqns (19) and (20), is continuous, $dJ/dz = 0$, as it should be. This is, of course, true even if the scattering length l_{sc} is energy dependent, $l_{\text{sc}} = l_{\text{sc}}(k)$, and including inelastic scattering processes. For elastic processes only, in the absence of capture, the current energy density is itself constant, $dG/dz = 0$.

In what follows, we shall confine ourselves to the case when the length l_{sc} does not depend on the electron energy. Moreover, we shall assume that l_{cp} is also energy-independent. The carrier concentration $n(z)$, defined by:

$$n(z) = \frac{2}{(2\pi)^3} \int f(z, \mathbf{k}) d\mathbf{k} = \frac{1}{\pi^2} \int_0^{\infty} f_0(z, k) k^2 dk, \quad (21)$$

satisfies (to within a constant factor) an equation identical to (17):

$$n(\zeta) = \frac{n^*}{2} E_2(\zeta) + \frac{l_{\text{tot}}}{2l_{\text{sc}}} \int_0^{\infty} n(\zeta') E_1(|\zeta - \zeta'|) d\zeta', \quad (22)$$

where $\zeta = z/l^*$ and $w = W/l^*$. Equation (22) is obtained by integrating eqn (17) over dk . Comparing eqns (17) and (22) we see that the distribution function and the concentration are related as follows:

$$f_0(z, k) = \frac{n(z)}{N_C} \exp\left(-\frac{E_k}{kT}\right), \quad (l_{sc} = \text{const}). \quad (23)$$

The electron current density is determined from eqns (19), (20) and (23):

$$J(\zeta) = -2v_R e \left(n^* E_3(\zeta) + \frac{l_{tot}}{l_{sc}} \int_0^w n(\zeta') \times E_2(|\zeta - \zeta'|) \text{sign}(\zeta - \zeta') d\zeta' \right). \quad (24)$$

Differentiating eqn (24) and using eqns (B3a) and (B5), we obtain the continuity equation:

$$\frac{dJ}{dz} = \frac{4v_R e n(z)}{l_{cp}}. \quad (25)$$

It follows from eqn (25) that the capture time τ_{cp} and length l_{cp} are related by $\tau_{cp} = l_{cp}/4v_R$. This relation is a consequence of the assumed independence of l_{cp} on energy, cf. Appendix C.

Below we present the results of a numerical integration of eqn (22) and compare them with an approximate analytical solution.

4.1. Exact solution

In eqn (22), the kernel $E_1(|\zeta - \zeta'|)$ has a logarithmic singularity at $\zeta' \rightarrow \zeta$. It is convenient to eliminate this singularity prior to a numerical discretization of eqn (22) by rewriting it in a different form:

$$\left(1 - \frac{l_{tot} [2 - E_2(\omega - \zeta) - E_2(\zeta)]}{2l_{sc}} \right) \frac{n(\zeta)}{n^*} = \frac{E_2(\zeta)}{2} + \frac{l_{tot}}{2l_{sc}} \int_0^w E_1(|\zeta - \zeta'|) \frac{[n(\zeta') - n(\zeta)]}{n^*} d\zeta'. \quad (26)$$

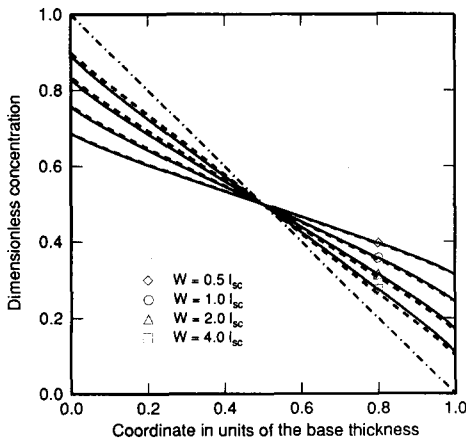


Fig. 2. Calculated concentration profile in the base. Solid lines: exact [eqn (26)]; dashed lines: approximation [eqns (33–35)]; stippled line: diffusion approximation.

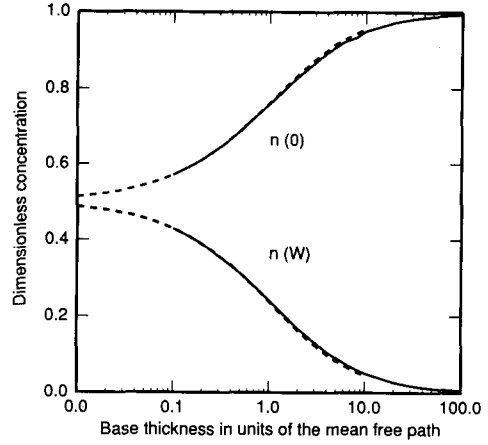


Fig. 3. Dependence of the calculated boundary values $n(0)$ and $n(W)$ of the dimensionless concentration (in units of n^*) on the dimensionless base thickness $w = W/l_{sc}$. Solid lines: exact solution of eqn (26); dashed lines: approximation [eqns (33–35)].

Figure 2 presents the calculated coordinate dependencies of the concentration for different base thicknesses W . For the time being, capture processes are neglected (i.e. we have set $l_{tot} = l_{sc}$). The dashed line corresponds to a purely diffusive approximation (valid in the limit $W \gg l_{sc}$, cf. Section 5), when $n(z) = n^*(1 - z/W)$.

As is evident from Fig. 2, the concentration profiles $n(z)$ are linear to a good approximation, even far away from the diffusion limit. Small deviations from linearity are seen only near the base boundaries. It is important to note, however, that the boundary values of the concentration depend strongly on the base thickness. The decrease of concentration at the base–emitter boundary corresponds to the fact that as the base thickness decreases, the diffusive transport goes over into a “ballistic” regime. In the latter limit, most of the electrons in the base move in the collector direction and $n(0) \approx n^*/2$. On the other hand, for $W \gg l_{sc}$ the distribution function becomes nearly symmetric as a result of scattering, and the concentration increases, $n(0) \rightarrow n^*$. The concentration at the base–collector junction varies in the opposite way: $n(W) \rightarrow n^*/2$ for $W \rightarrow 0$ and $n(W) \rightarrow 0$ for $W \rightarrow \infty$. All of these features of the distribution function can be ascertained directly from the solution of eqn (26)—by first expressing f_0 in terms of $n(z)$ with the help of eqn (23) and then substituting it into eqns (16). Both the coordinate and the angular dependencies of $f(z, k, u)$ can be evaluated in this way.

The calculated variations of $n(0)$ and $n(W)$ are plotted in Fig. 3 against the dimensionless base thickness $w \equiv W/l_{sc}$. We note that $n(0) + n(W) \approx n^*$ for all values of W . This result is intimately related to the good degree of linearity of $n(z)$ in the base, cf. Section 4.2.

Substituting the obtained $n(z)$ in eqn (24), we can evaluate the current. Expression (24) can be further simplified, using eqn (25):

$$J(z) = -2v_R e \left(\frac{n^*}{2} - \frac{1}{l_{sc}} \int_0^w n(z') \times E_2(z'/l_{tot}) dz' - \frac{2}{l_{cp}} \int_0^z n(z') dz' \right). \quad (27)$$

The dependence $J(W)$, calculated from (27), is shown in Fig. 4 by open circles. The fact that even for $w = 0.1$ the current does not yet reach its thermionic limit (en^*v_R) is owing to its weak dependence on w , which is shown below to be of the form $\propto(1 - w)$. It is interesting to compare the exact current with predictions of the diffusion model. According to the latter:

$$J_{diff} = -\frac{eD_0n^*}{W}, \quad (28)$$

where the diffusivity D_0 is related to the energy-independent mean free path by the usual expression (Ref. [9], p. 96):

$$D_0 = \frac{4}{3} l_{sc} v_R. \quad (29)$$

The standard diffusion-model result (28) is plotted in Fig. 4 by the dotted line; the dashed line corresponds to replacing n^*/W in eqn (28) by the exact gradient $[n(0) - n(W)]/W$, calculated from eqn (22). We see that the diffusion model substantially overestimates the current.

Figure 5 shows the ratio ζ of J to J_{diff} as a function of the base thickness. This ratio can be interpreted in a slightly different form, convenient for applications. The emitter efficiency γ :

$$\gamma = \frac{h}{1 + h}, \quad (30)$$

is defined in terms of the ratio h of the electron current J_{nB} , injected in the base, to the hole current J_{pE} injected in the emitter:

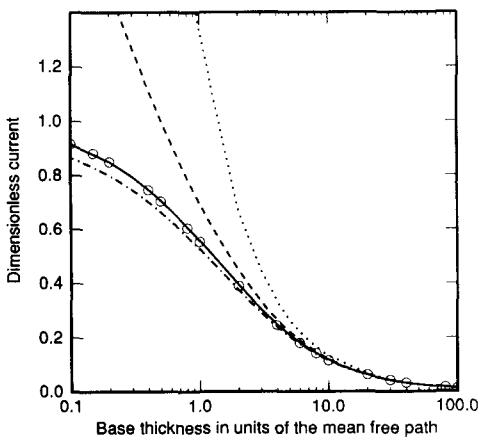


Fig. 4. Dependence of the calculated current density (in units of en^*v_R) on the dimensionless base thickness $w = W/l_{sc}$. Open circles: exact [eqn (27)]; solid line: approximation [eqns (36-37)]; dotted line: standard diffusion approximation [eqn (28)]; dashed line: diffusion approximation as in eqn (28) but with n^*/W replaced by an exactly calculated average concentration gradient; stippled line: the generalized diffusion approximation [eqn (45)].

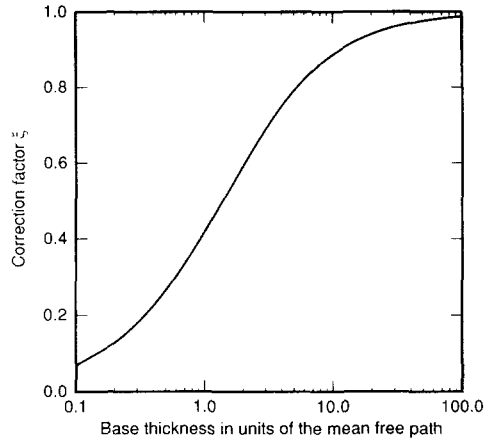


Fig. 5. Ratio ζ of the exact current to that calculated in the diffusion approximation [eqn (28)] as a function of the base thickness. Within the scale of the figure the curve is indistinguishable from that calculated with eqn (37).

$$h = \frac{\partial J_{nB}}{\partial J_{pE}} = \frac{J}{J_{diff}} \frac{\partial J_{diff}}{\partial J_{pE}} \approx \zeta h_{diff}, \quad (31)$$

where h_{diff} is the value of h , calculated in the diffusion model. The approximate relation in the right-hand side of (31) is obtained from eqns (27) and (28) if we assume the emitter to be sufficiently thick that the diffusion approximation be valid for the injected holes. A highly accurate analytical expression for $\zeta(w)$ is derived in the next Section [eqn (37)].

4.2. Approximate analytical solution

This solution is based on the observation that the dependence $n(z)$ is very close to linear, cf. Fig. 2. We can obtain, therefore, an iterative approximation for $n(z)$ by substituting into the integrand in the right-hand side of eqn (22) a linear interpolation of the form:

$$n(z) = n(0) - \frac{n(0) - n(W)}{W} z. \quad (32)$$

Performing the integration with the help of the formulae in Appendix B, we find:

$$n(\zeta) = \frac{1}{2} n^* E_2(\zeta) + \frac{1}{2} n(0) [2 - E_2(w - \zeta) - E_2(\zeta)] - \frac{n(0) - n(w)}{2w} [2\zeta - w E_2(w - \zeta) - E_3(w - \zeta) + E_3(\zeta)]. \quad (33)$$

Letting $\zeta = 0$ and $\zeta = w$ in eqn (33), we obtain a system of equations determining both $n(0)$ and $n(w)$, whence we find:

$$n(0) = \frac{n^*}{2} (1 + \delta(w)); \quad (34a)$$

$$n(w) = \frac{n^*}{2} (1 - \delta(w)), \quad (34b)$$

where

$$\delta(w) = \frac{w[1 - E_2(w)]}{1 + w - e^{-w}}. \quad (35)$$

Equation (33) with the values of $n(0)$ and $n(w)$, substituted from eqn (34), accurately describes the coordinate dependence of the concentration, including small deviations from the linearity near both boundaries, cf. Fig. 2.

Similarly, substituting (32) in the right-hand side of eqn (27) and using eqn (34), we obtain an approximate expression for the current that agrees to within 10% with the exact values plotted in Fig. 4. A still better approximation is obtained by substituting eqns (32) in eqn (24) and then *averaging* the resultant $J(z)$ over the base length [the current has a slight coordinate dependence because (32) is only an approximation to $n(z)$]. This procedure yields:

$$J = \frac{1}{W} \int_0^w J(z) dz = -\frac{4v_R en^*}{3w} \xi(w), \quad (36)$$

with the reduction factor $\xi(w)$ given by:

$$\begin{aligned} \xi(w) = 1 - & \frac{3[wE_4(w) + 1 - e^{-w}]}{4w} \\ & - \frac{3[1 - e^{-w} + wE_2(w)]}{4(1 + w - e^{-w})} \\ & \times \left(\frac{2}{3} - \frac{1 - e^{-w}}{w} + E_4(w) \right). \quad (37) \end{aligned}$$

The values of $n(0)$ and $n(w)$, calculated from eqns (34), are plotted in by Fig. 3 by the dashed lines. On the scale of the figure these curves are practically indistinguishable from the exact solutions. A similar agreement is obtained for the current, evaluated from eqns (36–37) and plotted in Fig. 4 by the solid line.

The above analytic expressions for δ and ξ can be used to investigate the asymptotic behavior of $n(0)$, $n(w)$ and J both for very long and very short base thicknesses. With the help of formulae in Appendix B, we find the following asymptotic expressions:

For $w \rightarrow 0$:

$$n(0) = \frac{n^*}{2} \left(1 + \frac{w}{2} [1 - C - \ln(w)] \right); \quad (38a)$$

$$n(w) = \frac{n^*}{2} \left(1 - \frac{w}{2} [1 - C - \ln(w)] \right); \quad (38b)$$

$$J(w) = -ev_R n^* (1 - w), \quad (38c)$$

where $C \approx 0.577$ is the Euler constant.

For $w \rightarrow \infty$:

$$n(0) = n^* \left(1 - \frac{1}{2w} \right); \quad (39a)$$

$$n(w) = \frac{n^*}{2w}; \quad (39b)$$

$$J(w) = -\frac{4ev_R n^*}{3w} \left(1 - \frac{1}{4w} \right). \quad (39c)$$

5. EFFECTIVE DIFFUSION COEFFICIENT

The fact that the exact concentration profile in the base turns out to be linear to a high degree, implies that it is possible to represent the current in the form:

$$J = eD \frac{dn}{dz}, \quad (40)$$

with an effective diffusion coefficient D that may be a function of n and ∇n .

Beginning with Persky[1] there have been several attempts to introduce such an effective diffusivity. It was usually assumed that the sought after formula should interpolate between the purely diffusive transport, $J = eD_0 dn/dz$, at low concentration gradients and the "thermionic" formula, $J = -env_R$, at high gradients. To satisfy the latter condition, the effective diffusivity must be a function of the *logarithmic* gradient $\nabla \ln n(r)$, at least in the thermionic limit. The simplest interpolation formulae with these properties are of the form:

$$D(\nabla \ln n) = D_0 \left[1 + \left(\frac{D_0}{v_R n} \left| \frac{dn}{dz} \right| \right)^m \right]^{-1/m}. \quad (41)$$

Persky's formula[1] has $m = 1$. Interpolation formulae of the form (41) have had little success. In our opinion, the main reason for their failure is that the product nv_R gives an incorrect flux in the thermionic limit. The correct flux, n^*v_R , is given by eqn (38c), and if we want to express it in terms of a physical concentration n in the base, $n(W) \ll n \ll n(0)$, then we must choose the thermionic current limit in the form $J = -en \cdot 2v_R$, cf. eqns (38a,b). This was already recognized by Berz[3] on the basis of a kinetic analysis of the transport near a perfect sink.

Since we have obtained an exceedingly accurate fit to the exact solution for the current density in a model that starts from a linear interpolation for the concentration, eqn (32), it is reasonable to define the effective diffusivity in terms of the ratio of the calculated current to the calculated concentration gradient. This gives:

$$D(w) = \frac{D_0 \xi(w)}{\delta(w)}, \quad (42a)$$

$$\frac{dn}{dz} = \frac{n(0) - n(W)}{W} = \frac{4v_R n^* \delta(w)}{3D_0 w}, \quad (42b)$$

where we have used eqns (29), (34) and (36). Eliminating w from eqns (42), we can express D as a function of $(1/n^*) dn/dz$. This function is plotted in Fig. 6 by the solid line. Moreover, we can look for an interpolation formula of the form similar to (41):

$$D \left(\frac{1}{n^*} \frac{dn}{dz} \right) = D_0 \left[1 + \left(\frac{D_0}{2v_R n^*} \left| \frac{dn}{dz} \right| \right)^m \right]^{-1/m}. \quad (43)$$

Interpolations with $m = 1, 2$ and 3 are shown in Fig. 6 by the broken lines. The least square fit gives $m = 2.01$. As seen from Fig. 6, the fit with $m = 2$ is quite accurate.

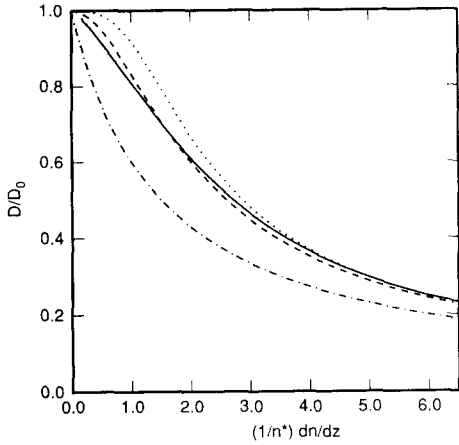


Fig. 6. Fit of the calculated generalized diffusivity $D(n^{*-1}\nabla n)$ (solid line) to interpolation formulae of the form (43). Stippled line: $m = 1$, dashed line: $m = 2$, dotted line: $m = 3$.

Formula (43) behaves correctly in both the diffusion and the thermionic limits. However, since D is not a function of the logarithmic derivative of n , this formula cannot be used directly in approximate calculations—unless one knows n^* . The best approximation in terms of the logarithmic derivative that can be inferred from (43) is to replace $n^*/2$ by n , keeping in mind that we have $n(0) + n(W) = n^*$ for all values of W , cf. eqns (34). Thus, we arrive at a generalized diffusion equation of the form:

$$J = \frac{eD_0}{\left[1 + \left(\frac{D_0}{2v_R} \frac{dn}{dz}\right)^2\right]^{-1/2}} \frac{dn}{dz} \quad (44)$$

Solution of eqn (44) is given by:

$$\left[\left(\frac{2ev_R n(z)}{|J|}\right)^2 - 1\right]^{1/2} + \arcsin\left(\frac{|J|}{2ev_R n(z)}\right) - \frac{\pi}{2} = \frac{2v_R(W-z)}{D_0}, \quad (45)$$

where we have assumed the boundary condition at the collector interface in the form $J = -2ev_R n(W)$. For a given current, eqn (45) describes the variation of $n(z)$ in the base. Conversely, if we assume a known carrier concentration $n(0)$ at the base-emitter interface, which may be obtained by solving the coupled transport and Poisson equations in the space-charge region, then eqn (45) yields the value of the current.

In the exact solution $n(0)/n^*$ is a function of the base thickness, or—equivalently—of the current. The diffusion equation (44) does not allow a simultaneous determination of both $n(0)$ and J . In this sense, the equation is rather useless if we restrict ourselves to the base region alone. However, using the generalized diffusion coefficient in a description of the entire system from the collector to the emitter (in a device modeling program) should give a better approximation to the concentration $n(0)$.

In order to test the consistency of such approach, we have evaluated J by substituting $n(0)$ in the form (34a) into eqn (45). The values of J thus obtained are plotted in Fig. 4 by the stippled line. We see that this procedure gives a vast improvement over the standard diffusion equation results. This offers hope that the generalized diffusion eqn (44) may find applications in device modeling programs.

6. HIGH-FREQUENCY RESPONSE

We shall assume that the periodic perturbation originates from an oscillatory variation of the base-emitter voltage:

$$V_{BE}(t) = V_{BE}^{(0)} + \delta V_{BE} e^{i\omega t} \quad (46)$$

and that the oscillating part is small in the sense that $\delta V_{BE} \ll kT/e$. The latter condition allows us to restrict ourselves to a harmonic perturbation of the boundary conditions (3):

$$\delta n^* = n_E \exp\left(-\frac{e(\Phi - V_{BE}^{(0)})}{kT}\right) \frac{e \delta V_{BE}}{kT} e^{i\omega t}. \quad (47)$$

In modern short-base transistors, the high frequency response is controlled by the time of electron propagation across the base and the capture processes are not important. However, in order to trace the transistor gain in the entire frequency range, we shall assume that at low frequencies, the gain is controlled by recombination of injected electrons in the base. For energy-independent capture length l_{cp} , the recombination time τ_{cp} is related to l_{cp} by $l_{cp} = 4v_R \tau_{cp}$, cf. Appendix C.

The time-dependent problem is solved by replacing the parameter l_{sc}/l_{tot} , that enters eqns (22) and (24), by:

$$\begin{aligned} \frac{l_{sc}}{l^*(E_k)} &= 1 + \frac{l_{sc}}{l_{cp}} + \frac{i\omega m l_{sc}}{\hbar k} \\ &= 1 + \frac{l_{sc}}{l_{cp}} + \frac{i\omega l_{sc}}{2v_R} \left(\frac{kT}{\pi E_k}\right)^{1/2}. \end{aligned} \quad (48)$$

The assumption of an energy-independent mean free path, used in the steady-state solution (Section 4), introduces no particular simplification in the time-dependent problem, since the length l^* depends on the energy even when both l_{sc} and l_{cp} do not. Thus, eqn (23) is no longer valid even for $l_{sc} = \text{const}$. We must, therefore, solve eqn (17) for each E_k and determine the concentration by the following relation:

$$n(z) = \frac{2N_C}{\sqrt{\pi(kT)^3}} \int_0^z f_0(z, E_k) \sqrt{E_k} dE_k. \quad (49)$$

Similarly, having determined $f_0(z, E_k)$, we can find $G(z, E_k)$ with the help of eqn (20). The total current is then determined by eqn (19).

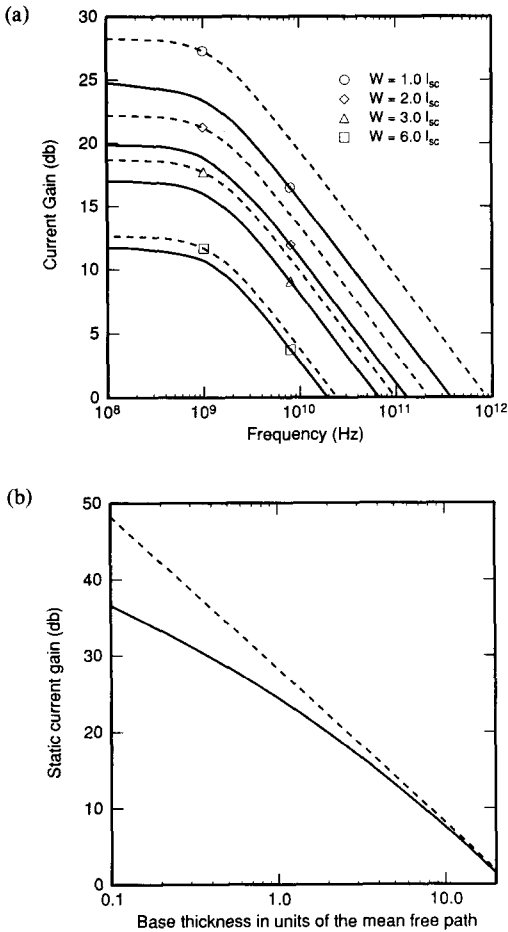


Fig. 7. Calculated current gain β . The assumed capture length $l_{cp} = 10^3 l_{sc}$, scattering length $l_{sc} = 500 \text{ \AA}$, and thermionic velocity $v_R = 10^7 \text{ cm s}^{-1}$. Dashed lines correspond to the usual diffusion approximation. (a) Frequency dependence $\beta(f)$ for several base thicknesses; and (b) base thickness dependence of the static gain $\beta(0)$.

Figure 7 presents the current gain β , defined as:

$$\beta = \frac{\partial I_C}{\partial I_B} \quad (50)$$

and calculated assuming a unity emitter efficiency and neglecting capacitive effects as well as recombination in the space charge region. The frequency dependence $\beta(f)$ is shown in Fig. 7a for several base thicknesses. Dashed lines indicate the usual diffusion approximation to β :

$$\beta_{diff} = \frac{1}{2 \sinh^2[(1 + i\omega\tau_{cp})^{1/2} W / (2(D_0\tau_{cp})^{1/2})]} \quad (51)$$

We see from Fig. 7a that for $W \sim l_{sc}$ the diffusion approximation overestimates the cutoff frequency f_T by about a factor of 2. The static gain $\beta(0)$ is plotted in Fig. 7b against the dimensionless base thickness w . For $w \gg 1$ the gain is proportional to w^{-2} , which is characteristic of the diffusion approximation. In the short-base range the dependence is slower, approaching w^{-1} in the limit $w \rightarrow 0$.

7. CONCLUSION

We have analyzed the transport of minority carriers in the base of a BJT. The analysis is based on the exact solution of the Boltzmann transport equation. A key ingredient of our approach is the formulation of the boundary conditions for the distribution function, eqns (3) and (7). The solution is presented both for the steady-state and the small-signal oscillatory problems and is compared with the standard diffusion equation results.

For the steady-state problem, we have derived an analytic approximation [eqns (33–37)] to the exact solution, which gives highly accurate results for arbitrary base thicknesses. The analytic solution is asymptotically exact in the limit of ultra-short base lengths [eqns (38)], where the minority-carrier transport can be regarded as thermionic, as well as in the long-base limit [eqns (39)], where the diffusion equation is rigorously valid.

On the basis of our analytic solution, we propose and test a new expression [eqn (44)] for the effective (concentration-dependent) diffusion flux, that interpolates between the diffusive and thermionic limits. It should be emphasized that no diffusion model can successfully describe the short-base transport, if one is confined to the consideration of the base region alone. Indeed, our results show that it is impossible in principle to impose conditions on the carrier concentration at the base junctions, since the boundary values of the concentration themselves depend on the transport in the base. This invalidates any model of transport in a short base (including generalized diffusion, hydrodynamic models, etc.) that require boundary conditions on the concentration. It is possible to treat the base transport in a self-contained fashion only within an approach based on the kinetic equation, where one can impose conditions on the *inbound electron fluxes* at the boundaries, as we have done in eqns (3) and (7). Phenomenological models based on the drift-diffusion equation can be useful provided the semiconductor region under consideration extends beyond the base—both into the emitter and the collector layers. A generalized diffusion equation (44) involving the effective diffusivity may find applications in numerical device modeling.

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APPENDIX A

Microscopic Expressions for the Scattering Integrals and the Mean Free Path

For electron scattering by acoustic phonons (in the quasi-elastic approximation) and by non-polar optic phonons, the outgoing collision integral is of the form:

$$\frac{\hbar k}{m} \hat{S}_k^{(-)} f = \frac{f_0(\mathbf{r}, k)}{l_{sc}(k)}, \tag{A1}$$

with

$$\frac{1}{l_{sc}(k)} = \frac{1}{l_{ac}} + \frac{1}{l_{op}} \left(\frac{\exp(\hbar\omega_{op}/2kT)\sqrt{E_k - \hbar\omega_{op}} + \exp(-\hbar\omega_{op}/2kT)\sqrt{E_k + \hbar\omega_{op}}}{2 \cosh(\hbar\omega_{op}/2kT)\sqrt{E_k}} \right), \tag{A2}$$

where

$$\frac{1}{l_{ac}} = \frac{E_{ac}^2 m^2 k T}{\pi \rho s^2 \hbar^4}, \tag{A3a}$$

$$\frac{1}{l_{op}} = \frac{E_{op}^2 m^2 \hbar \omega_{op} \coth(\hbar\omega_{op}/2kT)}{2\pi \rho s^2 \hbar^4}, \tag{A3b}$$

E_{ac} and E_{op} are the deformation potentials for acoustic and optic displacements, respectively, $\hbar\omega_{op}$ is the optical phonon energy, ρ the material density and s the sound velocity.

The incoming collision integral is of the form:

$$\frac{\hbar k}{m} \hat{S}_k^{(+)} f = \frac{f_0}{l_{ac}} + \frac{1}{l_{op}} \left(\frac{\exp(\hbar\omega_{op}/2kT)\sqrt{E_k + \hbar\omega_{op}} f_0(E_k + \hbar\omega_{op}) + \exp(-\hbar\omega_{op}/2kT)\sqrt{E_k - \hbar\omega_{op}} f_0(E_k - \hbar\omega_{op})}{2 \cosh(\hbar\omega_{op}/2kT)\sqrt{E_k}} \right). \tag{A4}$$

In the case when scattering is determined by the interaction with impurity centers of charge Ze and concentration N_{imp} , the collision integral is of the form:

$$\left(\frac{\partial f_k}{\partial t} \right)_{imp} = \frac{N_{imp} Z^2 e^4}{2(2m)^{1/2} \epsilon_0^2 E_k^{2/3}} \int \frac{(f_k - f_{k'}) d\Omega_{k'}}{[2\gamma + 1 - \cos(\mathbf{k} \wedge \mathbf{k}')]^2}, \tag{A5}$$

Here ϵ_0 is the dielectric permittivity, $\Omega_{k'}$ is the solid angle in \mathbf{k}' space, $\mathbf{k} \wedge \mathbf{k}'$ is the angle between vectors \mathbf{k} and \mathbf{k}' of the same magnitude k , $\cos(\mathbf{k} \wedge \mathbf{k}') \equiv \mathbf{k} \cdot \mathbf{k}' / k^2$ and γ is the screening parameter:

$$\gamma = \frac{\kappa^2 \hbar^2}{8mE_k}, \tag{A6}$$

where κ is the inverse screening length. Inasmuch as γ is usually small, $\gamma \ll 1$, the incoming collision integral is strongly anisotropic. With such a collision integral, the Boltzmann equation is much more difficult to solve than for isotropic scattering. It is possible, however to introduce an effective scattering term that produces the same results for the low-field diffusivity as the exact expression (A5), and, at the same time, is isotropic. To do this, we must modify both the incoming and the outgoing collision terms. This procedure is sketched below. It results in an approximate expression for the incoming collision integral that is of the form (13).

Assume that the effective collision integral is of the form:

$$\left(\frac{\partial f_k}{\partial t} \right)_{imp} = \frac{N_{imp} Z^2 e^4}{2(2m)^{1/2} \epsilon_0^2 E_k^{3/2}} \int R(\mathbf{k}; \mathbf{k}') [f_0(E_k) - f_k] d\Omega_{k'} \tag{A7}$$

and choose $R(\mathbf{k}; \mathbf{k}')$ in such a way that both the approximate expression (A7) and the exact (A5) produce the same result

if the distribution function is given by:

$$f_k = f_0(E_k) + \chi(E_k) \cos(\mathbf{k} \wedge \mathbf{z}), \tag{A8}$$

with an arbitrary function $\chi(E_k)$. This requirement is satisfied if:

$$\int_{-1}^1 \left(R(\mathbf{k}; \mathbf{k}') - \frac{1 - \cos(\mathbf{k} \wedge \mathbf{k}')}{[2\gamma + 1 - \cos(\mathbf{k} \wedge \mathbf{k}')]^2} \right) \times d[\cos(\mathbf{k} \wedge \mathbf{k}')] = 0, \tag{A9}$$

as can be easily verified by substituting (A8) into (A5) and (A7). Choosing $R(\mathbf{k}; \mathbf{k}')$ in the symmetric form:

$$R(\mathbf{k}; \mathbf{k}') = R(\mathbf{k} \wedge \mathbf{k}') = \frac{1 - \cos(\mathbf{k} \wedge \mathbf{k}')}{[2\gamma + 1 - \cos(\mathbf{k} \wedge \mathbf{k}')]^2}, \tag{A10}$$

substituting (A10) into (A7), and performing the integration, we find:

$$\left(\frac{\partial f_k}{\partial t} \right)_{imp} = \frac{f_0(E_k) - f_k}{\tau(E_k)}, \tag{A11}$$

where $\tau(E_k)$ is the usual scattering time due to interaction with charged impurities:

$$\frac{m}{\hbar k} \frac{1}{\tau(E_k)} \equiv \frac{1}{l_{imp}(E_k)} = \frac{\pi N_{imp} Z^2 e^4}{2 \epsilon_0^2 E_k^2} \times \left[\ln \left(\frac{1+\gamma}{\gamma} \right) + \frac{1}{1+\gamma} \right]. \tag{A12}$$

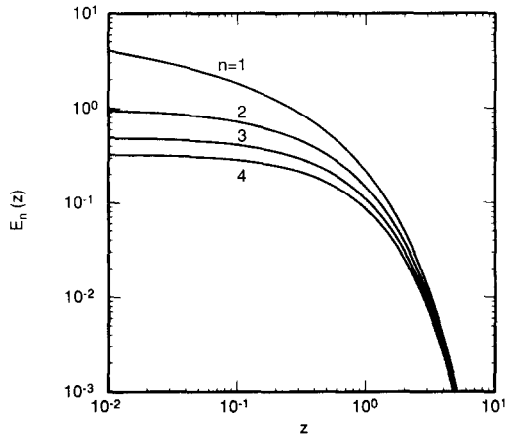


Fig. B1. The exponential integrals $E_n(z)$.

APPENDIX B

Properties of Exponential Integrals, Eqn (18)

Exponential integrals $E_n(z)$ are defined by:

$$E_n(z) = \int_1^\infty e^{-zt} t^{-n} dt = \int_0^1 t^{n-2} e^{-zt} dt. \tag{B1}$$

Figure 1 plots $E_n(z)$ for $n = 1, \dots, 4$. The function $E_1(z)$ is related to the exponential integral $Ei(z)$ by the following expression (see Ref. [11], pp. xxxii and 925):

$$E_1(z) = -Ei(-z) - C - \ln z - \sum_{k=1}^\infty \frac{(-z)^k}{kk!}, \quad z > 0, \tag{B2}$$

where $C = 0.577\dots$ is the Euler constant. Properties of $E_n(z)$ can be derived from those of $Ei(z)$ using the recursive relations:

$$\frac{dE_{n+1}(z)}{dz} = -E_n(z); \tag{B3a}$$

$$nE_{n+1}(z) = e^{-z} - zE_n(z). \tag{B3b}$$

Relations and integrals used in this present work are listed below:

$$E_{n+1}(0) = \frac{1}{n}. \tag{B4}$$

$$\frac{dE_{n+1}(|z-z'|)}{dz'} = E_n(|z-z'|)\text{sign}(z-z'). \tag{B5}$$

$$\int_0^w E_n(|z-z'|) dn' = \frac{2}{n} - E_{n+1}(w-z) - E_{n+1}(z); \tag{B6a}$$

$$\int_0^w (z-z')E_n(|z-z'|) dz' = (w-z)E_{n+1}(w-z) - zE_{n+1}(z) + E_{n+2}(w-z) - E_{n+2}(z); \tag{B6b}$$

$$\begin{aligned} &\int_0^w (z-z')^2 E_n(|z-z'|) dz' \\ &= \frac{4}{n+2} - z^2 E_{n+1}(z) - 2z E_{n+2}(z) - 2E_{n+3}(z) \\ &\quad - (w-z)^2 E_{n+1}(w-z) - 2(w-z) E_{n+2}(w-z) \\ &\quad - 2E_{n+3}(w-z). \end{aligned} \tag{B6c}$$

$$\begin{aligned} &\int_0^w E_n(|z-z'|)\text{sign}(z-z') dz' \\ &= E_{n+1}(w-z) - E_{n+1}(z); \end{aligned} \tag{B7a}$$

$$\begin{aligned} &\int_0^w (z-z')E_n(|z-z'|)\text{sign}(z-z') dz' \\ &= \int_0^w (|z-z'|)E_n(|z-z'|) dz' = \frac{2}{n+1} - zE_{n+1}(z) \\ &\quad - E_{n+2}(z) - (w-z)E_{n+1}(w-z) - E_{n+2}(w-z). \end{aligned} \tag{B7b}$$

For large values of z , it is convenient to use the asymptotic expansion:

$$\begin{aligned} E_n(z) &\sim e^{-z} \sum_{k=1}^\infty \frac{(-1)^{k+1}(n+k-2)!}{(n-1)!z^k} \\ &\approx \frac{1}{z} e^{-z}, \quad (z \rightarrow \infty). \end{aligned} \tag{B8}$$

APPENDIX C

Transition to the Diffusion Equation

Let us illustrate how the integral eqn (22) goes over into the usual diffusion equation in the limit when $W \gg l_{sc}$. The kernel $E_1(|\zeta - \zeta'|)$ diverges logarithmically as $\zeta \rightarrow \zeta'$, while for $|\zeta - \zeta'| \gg 1$ it decreases exponentially. Inasmuch as the concentration $n(\zeta')$ varies little on the mean-free-path length, it can be expanded about ζ to within terms of order $(\zeta - \zeta')$. The linear term vanishes upon integration. Neglecting the terms of order $E_n(w - \zeta)$ and $E_n(\zeta)$ and integrating with the help of formulae in Appendix B, we obtain:

$$\frac{1}{3} l_{sc} \frac{\partial^2 n}{\partial z^2} = \frac{1}{l_{cp}} n(z). \tag{C1}$$

In writing down eqn (22), we had assumed that the recombination length l_{cp} was independent of energy and $l_{cp} \gg l_{sc}$. For recombination on centers with a capture cross-section σ_{cp} , the recombination length equals $l_{cp} = 1/N\sigma_{cp}$, where N is the concentration of recombination centers. This implies that the capture cross-section σ_{cp} must also be taken energy-independent. Therefore, the average time of capture is determined by the relation $\tau_{cp}^{-1} = \bar{v}\sigma_{cp}N$, where \bar{v} is the mean value of the absolute magnitude of electron velocity. For the Maxwellian distribution, $\bar{v} = 4v_R$, whence

$$l_{cp} = 4v_R \tau_{cp}. \tag{C2}$$

Substituting this expression in eqn (C1) and taking into account eqn (29), we arrive at the usual diffusion equation.