MMSE Estimation of Nonlinear Parameters of Multiple Linear/Quadratic Chirps

Hsiang-Tsun Li and Petar M. Djurić

***Abstract*—An iterative algorithm similar to our MMSE estimation procedure proposed in another paper by the authors is applied to parameter estimation of multiple superimposed chirp signals in white Gaussian noise. The necessary parameter initializations of the procedure are accomplished by peak detections in the Choi–Williams time–frequency distribution of the data followed by application of the least-squares principle. A comparison between our scheme and the alternating projection (AP) method is demonstrated by computer simulation.**

I. Introduction

Chirp signals are frequently encountered in many scientific and engineering areas. In sonar applications, radar, and physics, the observed measurements are often modeled as amplitude modulated chirp signals embedded in Gaussian noise. Many procedures have been proposed for solving this problem, most of them based on the maximum likelihood (ML) principle or rank reduction techniques. Much of the research over the past ten years has focused on the single chirp signal, [1], [4], and not until recently has the problem related to multiple superimposed chirps been investigated [3], [5]. However, the disadvantages of the above estimators are that they can either process only one signal and/or need fairly high signal-to-noise ratios (SNR's) to have good performance.

In [6], we proposed a minimum mean square error (MMSE) method for estimating the parameters of damped sinusoids. In this correspondence, we extend the MMSE method proposed there to chirp signals and show that it has excellent performance for even low SNR's and short data records. The method exploits the shape of the posterior distribution of the chirp parameters and operates iteratively so that it processes only one chirp signal at a time. Thus, the enormous computational burden of multidimensional integrations is avoided, and the computational load is thereby drastically reduced.

An important issue in the implementation of the procedure is its initialization. We propose an efficient initialization technique that consists of two steps. First, the initial frequency estimates are obtained from the data's Choi–Williams distribution. Then, the initial values of the corresponding chirp rates are found by a least-squares algorithm. Once these values are determined, the starting estimates of the remaining nonlinear parameters of the strongest chirp signal are estimated, followed by parameter estimation of the second strongest signal, and so on.

II. Problem Statement

Let $y[n], n \in Z_N = \{n_0, n_0+1, \cdots, n_0+2N\}$ be a set of $2N+1$ observed samples composed of q complex quadratic chirps corrupted by an additive white Gaussian noise process. That is

$$y[n] = \sum_{k=1}^{q} a_k[n] \exp\{j(\beta_k n^3 + \alpha_k n^2 + \omega_k n + \phi_k)\} + w[n] \quad (1)$$

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where $a_k[n], \beta_k, \alpha_k, \omega_k, \phi_k$ are the real amplitude, quadratic chirp rate, linear chirp rate, initial frequency, and phase of the kth signal, respectively, and $j = \sqrt{-1}$. The random samples $w[n]$ are complex Gaussian, independent, and identically distributed whose real and imaginary components have zero mean and unknown variance $\sigma^2/2$. The number of chirps q is assumed to be *known*. Suppose that the amplitude of the kth signal can be decomposed by p_k basis functions, or

$$a_k[n] = \sum_{l=0}^{p_k-1} a_{k,l}\eta_{k,l}[n] \quad (2)$$

where $a_{k,l}$ is the amplitude associated with the lth basis function $\eta_{k,l}$ of the kth signal. The amplitudes $a_{k,l}$ are unknown, whereas the basis functions $\eta_{k,l}$ are *known*.

The received data vector $\boldsymbol{y}$ can be written in a vector-matrix form according to

$$\boldsymbol{y} = \boldsymbol{H}(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta})\boldsymbol{a} + \boldsymbol{w} \quad (3)$$

where $\boldsymbol{H}$ is a $(2N+1) \times L$ matrix whose columns span the signal space, $L = \Sigma_{k=1}^{q} p_k \leq (2N+1)$, $\boldsymbol{a}$ is an $L \times 1$ vector of complex amplitudes, and $\boldsymbol{w}$ is an $L \times 1$ noise vector with $\boldsymbol{w} \sim \mathcal{CN}(\boldsymbol{0}, \sigma^2\boldsymbol{I})$. The vector $\boldsymbol{a}$ is comprised of q concatenated amplitude vectors, which correspond to the superimposed signals

$$\boldsymbol{a}^T = [\boldsymbol{a}_1^T \quad \boldsymbol{a}_2^T \quad \cdots \boldsymbol{a}_q^T]$$

where

$$\boldsymbol{a}_k^T = [a_{k,0}e^{j\phi_k} \quad a_{k,1}e^{j\phi_k} \quad \cdots \quad a_{k,p_k-1}e^{j\phi_k}]$$

and the matrix $\boldsymbol{H}(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is composed of q submatrices

$$\boldsymbol{H}(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = [\boldsymbol{H}_1(\omega_1, \alpha_1, \beta_1)\boldsymbol{H}_2(\omega_2, \alpha_2, \beta_2) \cdots \boldsymbol{H}_q(\omega_q, \alpha_q, \beta_q)]$$

where the $\boldsymbol{H}_k$'s are $(2N+1) \times p_k$ matrices $k = 1, 2, \cdots, q$

$$\boldsymbol{H}_k(\omega_k, \alpha_k, \beta_k) = [\boldsymbol{d}_{k,0}(\omega_k, \alpha_k, \beta_k)\boldsymbol{d}_{k,1}(\omega_k, \alpha_k, \beta_k) \cdots \boldsymbol{d}_{k,p_k-1}(\omega_k, \alpha_k, \beta_k)]$$

and

$$\begin{aligned}\boldsymbol{d}_{k,l}(\omega_k, \alpha_k, \beta_k)^T = [&\eta_{k,l}[n_0] \exp\{\beta_k n_0^3 + \alpha_k n_0^2 + \omega_k n_0\} \\ &\cdots \eta_{k,l}[n_0+2N] \exp\{\beta_k(n_0+2N)^3 \\ &+ \alpha_k(n_0+2N)^2 + \omega_k(n_0+2N)\}].\end{aligned}$$

Given the observations $\boldsymbol{y}, q, p_k$, and $\eta_{k,l}$, the objective is to obtain the MMSE estimates $\hat{\omega}_k, \hat{\alpha}_k$, and $\hat{\beta}_k, k = 1, 2, \cdots, q$.

III. MMSE Estimator

The MMSE estimate of ω_k is given by the following $3q$-dimensional integral

$$\hat{\omega}_k = \int_{\{\boldsymbol{\omega},\boldsymbol{\alpha},\boldsymbol{\beta}\}} \omega_k f(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{y})\, d\boldsymbol{\beta}\, d\boldsymbol{\alpha}\, d\boldsymbol{\omega} \quad (4)$$

where $f(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{y})$ is the posterior probability density function (pdf) of the signal parameters $\boldsymbol{\beta}, \boldsymbol{\alpha}$, and $\boldsymbol{\omega}$. The MMSE estimates of α_k and β_k are similarly defined.

The posterior pdf's are highly concentrated functions around the true values of $\boldsymbol{\omega}, \boldsymbol{\alpha}$, and $\boldsymbol{\beta}$. Then, if i denotes the current iteration of our estimator, and $\hat{\omega}_k^{(i)}, \hat{\alpha}_k^{(i)}$ and $\hat{\beta}_k^{(i)}$ are the current estimates of

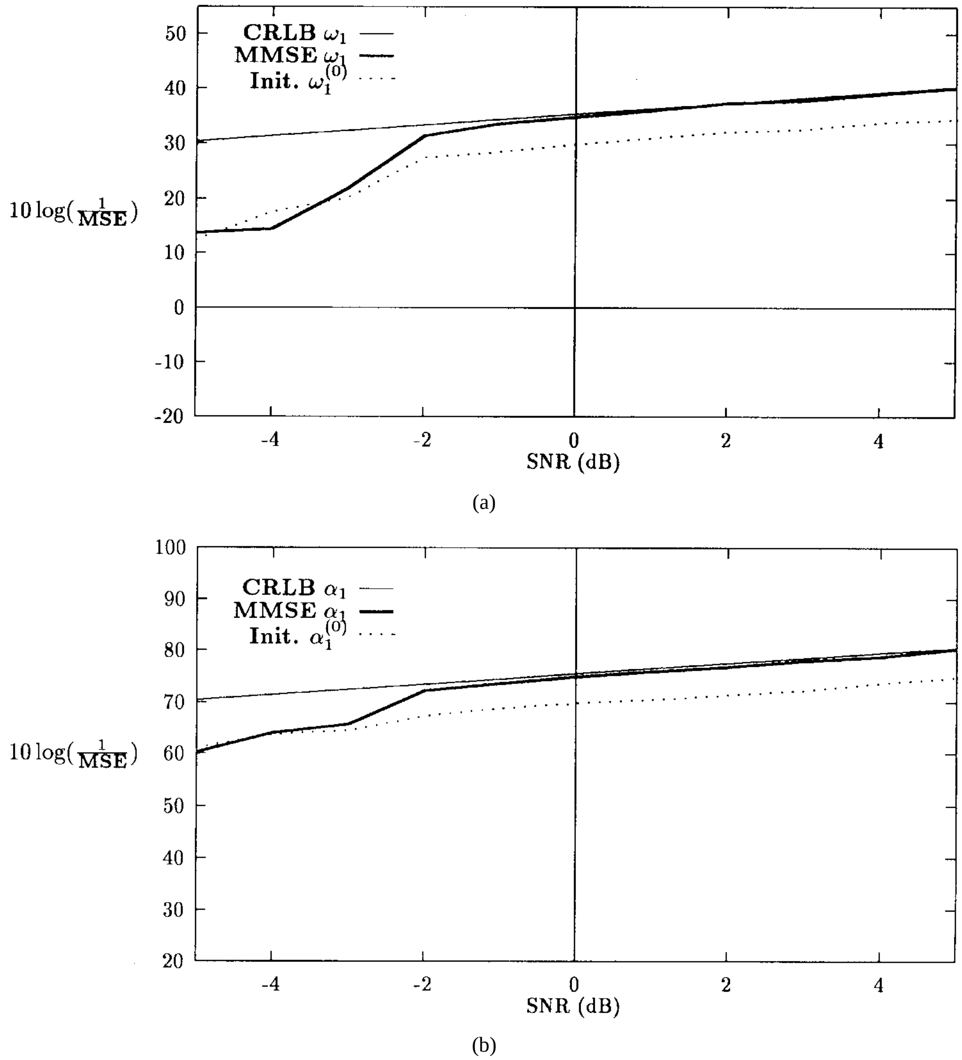


Fig. 1. Performance comparison of the MMSE and initial parameter estimates for $q = 2$ (a) $\omega = 0.06283$, (b) $\alpha = 0.0003142$.

ω_k, α_k, and β_k, respectively, $k = 1, 2, \cdots, q$, we can approximate the posterior density $f(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{y})$ by

$$f(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{y}) \simeq f(\omega_k, \alpha_k, \beta_k|\boldsymbol{y}, \hat{\boldsymbol{\omega}}_{(-k)}^{(i)}, \hat{\boldsymbol{\alpha}}_{(-k)}^{(i)}, \hat{\boldsymbol{\beta}}_{(-k)}^{(i)}) \cdot \prod_{\substack{l=1 \\ l \neq k}}^{q} \delta(\omega_l - \hat{\omega}_l^{(i)}, \alpha_l - \hat{\alpha}_l^{(i)}, \beta_l - \hat{\beta}_l^{(i)}) \tag{5}$$

where $\hat{\boldsymbol{\omega}}_{(-k)}^{(i)}, \hat{\boldsymbol{\alpha}}_{(-k)}^{(i)}$, and $\hat{\beta}_{(-k)}^{(i)}$ denote the estimates of the ith iteration of all the frequencies and linear and quadratic chirp rates except the ones of the kth signal. With this approximation, our $3q$-dimensional integral in (4) reduces to q individual three-dimensional (3-D) integrals, that is

$$\hat{\omega}_k^{(i)} = \int_{\{\omega_k, \alpha_k, \beta_k\}} \omega_k f(\omega_k, \alpha_k, \beta_k|\boldsymbol{y}, \hat{\boldsymbol{\omega}}_{(-k)}^{(i)} \cdot \hat{\boldsymbol{\alpha}}_{(-k)}^{(i)}, \hat{\boldsymbol{\beta}}_{(-k)}^{(i)})\, d\beta_k\, d\alpha_k\, d\omega_k. \tag{6}$$

The estimates of $\hat{\alpha}_k$ and $\hat{\beta}_k$ are obtained analogously. Theefore, instead of solving (4), we compute integrals of the form given by (6). The estimates of the unknown parameters are found iteratively, one at a time. The integrals in (6) can be solved efficiently by the adaptive Gaussian quadrature (AGQ) technique proposed in [6].

IV. Initialization Procedure

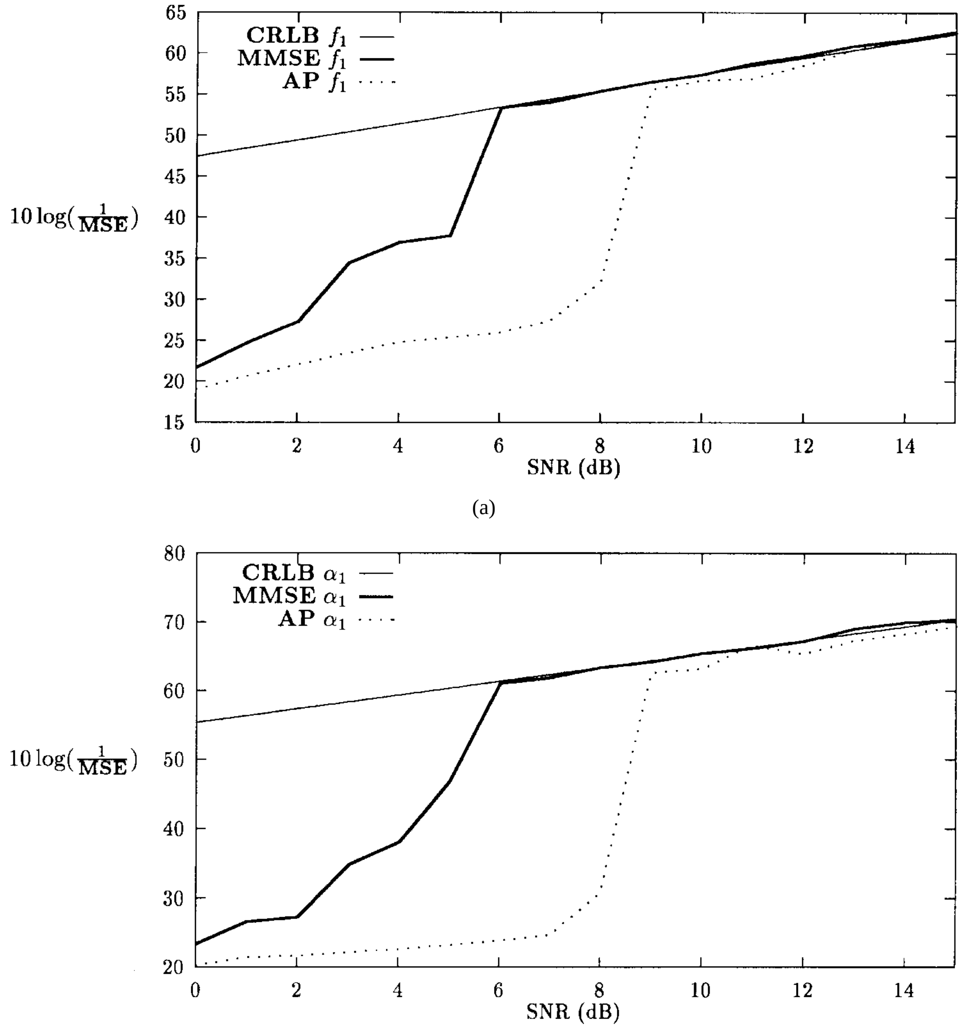
To start the iterative algorithm, we need to determine the initial estimates $(\hat{\beta}_k^{(0)}, \hat{\alpha}_k^{(0)}, \hat{\omega}_k^{(0)})$ of $(\beta_k, \alpha_k, \omega_k), k = 1, 2, \cdots, q$. First, we outline the initialization procedure for linear chirp signals, i.e., for signals with $\beta_k = 0$, and then extend it to quadratic chirp signals. Since the instantaneous frequency for the kth signal at the instant n is

$$\tilde{\omega}_k[n] = 2\alpha_k n + \omega_k, \qquad k = 1, 2 \cdots q \tag{7}$$

we can express the value of ω_k in terms of α_k as

$$\omega_k = \tilde{\omega}_k[n] - 2\alpha_k n. \tag{8}$$

Using (8), we propose an initialization that consists of two steps. The first step is the estimation of the kth signal's instantaneous frequency at $n = n_0 + N, \hat{\tilde{\omega}}_k[n_0 + N]$ by detecting the kth peak in the Choi–Williams time–frequency distribution of the data. We choose the instant $n = n_0 + N$ because the time–frequency distribution has the smallest variance at $n = n_0 + N$. Since the Choi–Williams distribution concentrates on the chirp's instantaneous frequency, we search along the frequency axis at $n = n_0 + N$ to find the frequencies corresponding to the peaks of the distribution and obtain the estimates $\hat{\tilde{\omega}}_k[n_0 + N], k = 1, 2, \cdots q$. Once the instantaneous frequency estimates for each signal at $n = n_0 + N$ are determined, we express the initial frequencies in terms of α_k by (8). Since the discrete Choi–Williams distribution is periodic with a



(a)

(b)

Fig. 2. Performance comparison of the MMSE and alternating projection (AP) estimators for $q = 2, N = 21, a_1 = a_2 = 1.0$. (a) $f_1 = 0.2$. (b) $\alpha = 0.01$.

period of π instead of 2π, the initial estimates are given either by $\hat{\tilde{\omega}}_k[n_0+N] - 2\hat{\alpha}_k^{(0)}(n_0+N)$ or $\hat{\tilde{\omega}}_k[n_0+N] + \pi - 2\hat{\alpha}_k^{(0)}(n_0+N)$. The decision as to which one will represent the initial estimate is made in the second step of the initialization, which involves a search for best initial estimates of the chirp rates.

The second step is based on the following criterion:

$$[\hat{\omega}_k^{(0)}, \hat{\alpha}_k^{(0)}] = \arg \max_{\omega \in \mathcal{F}_{k,\alpha}} \boldsymbol{y}^H \boldsymbol{P}(\omega, \alpha, \boldsymbol{\theta}_{k-1})\boldsymbol{y} \qquad k = 1, 2, \cdots, q \tag{9}$$

where $\mathcal{F}_k = \{\hat{\tilde{\omega}}_k[n_0+N] - 2\alpha(n_0+N), \hat{\tilde{\omega}}_k[n_0+N] + \pi - 2\alpha(n_0+N)\}, \boldsymbol{\theta}_{k-1} = [\hat{\omega}_1^{(0)}, \hat{\alpha}_1^{(0)}, \cdots, \hat{\omega}_i^{(0)}, \hat{\alpha}_i^{(0)}, \cdots, \hat{\omega}_{k-1}^{(0)}, \hat{\alpha}_{k-1}^{(0)}]$, and $\boldsymbol{\theta}_0 = \emptyset$ (empty set), and $\boldsymbol{P}(\omega, \alpha, \boldsymbol{\theta}_{k-1})$ is a projection matrix defined by

$$\boldsymbol{P}(\omega, \alpha, \boldsymbol{\theta}_{k-1}) = \boldsymbol{H}(\omega, \alpha, \boldsymbol{\theta}_{k-1})(\boldsymbol{H}^H(\omega, \alpha, \boldsymbol{\theta}_{k-1}) \cdot \boldsymbol{H}(\omega, \alpha, \boldsymbol{\theta}_{k-1}))^{-1}\boldsymbol{H}^H(\omega, \alpha, \boldsymbol{\theta}_{k-1}).$$

The chirp rate of the strongest component is estimated first, followed by the estimates of the chirp rate corresponding to the second strongest signal, and so on.

The extension of the above procedure to quadratic chirp signals is straightforward. We assume that the instantaneous frequencies for each signal around the instant $n = n_0 + N$ are well separated and change smoothly with n. Since the instantaneous frequency for the kth signal is

$$\tilde{\omega}_k[n] = 3\beta_k n^2 + 2\alpha_k n + \omega_k, \qquad k = 1, 2 \cdots q \tag{10}$$

we can express ω_k and α_k in terms of β_k as

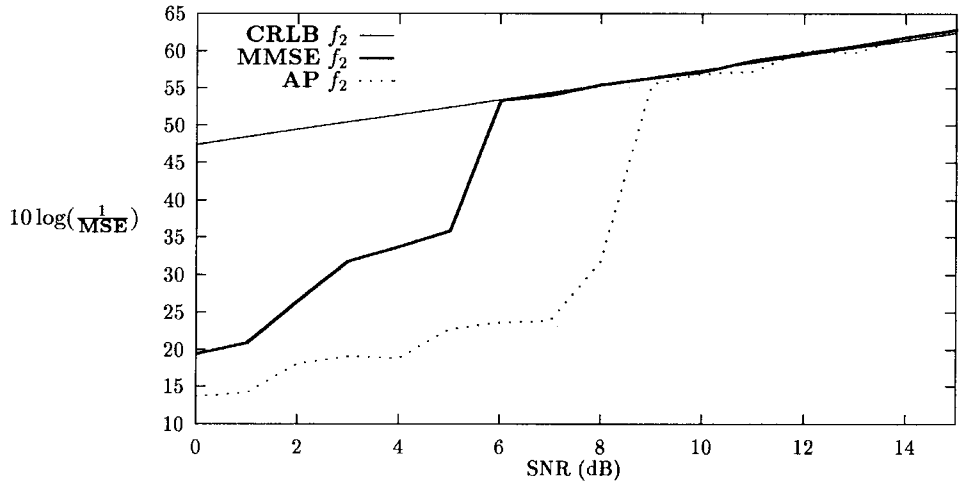
$$\alpha_k = \tfrac{1}{4}(\tilde{\omega}_k[n+1] - \tilde{\omega}_k[n-1] - 12n\beta_k) \tag{11}$$

and

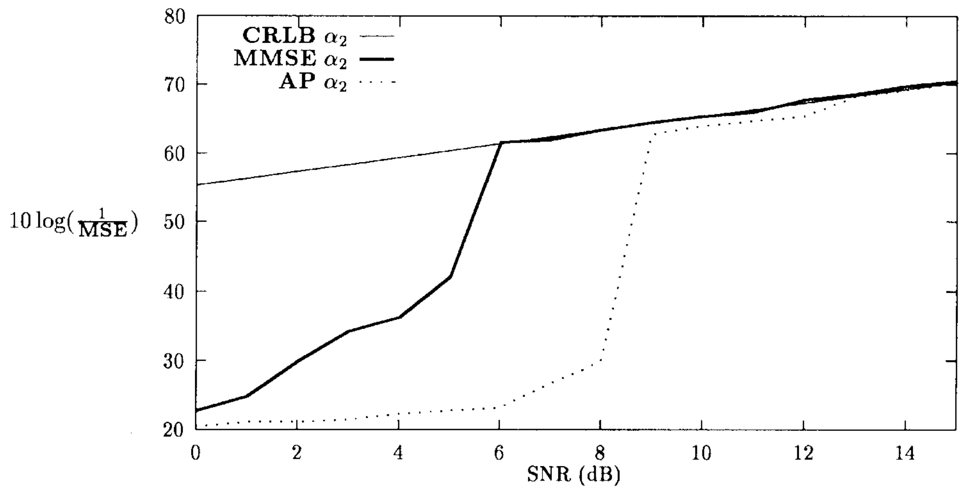
$$\begin{aligned}\omega_k &= \tilde{\omega}_k[n] - 3\beta_k n^2 - 2\alpha_k n \\ &= \tilde{\omega}_k[n] - 3\beta_k n^2 - \tfrac{1}{2}(\tilde{\omega}_k[n+1] - \tilde{\omega}_k[n-1] - 12n\beta_k).\end{aligned} \tag{12}$$

Again, we first estimate the instantaneous frequencies $\tilde{\omega}_k[n_0+N], k = 1, 2, \cdots q$ from the largest peaks of the Choi–Williams distribution. Similarly, we estimate $\tilde{\omega}_k[n_0+N+1]$ and $\tilde{\omega}_k[n_0+N-1]$ so that these estimates satisfy $|\hat{\tilde{\omega}}_k[n_0+N+1] - \hat{\tilde{\omega}}_k[n_0+N]| < |\hat{\tilde{\omega}}_k[n_0+N+1] - \hat{\tilde{\omega}}_i[n_0+N]|$ and $|\hat{\tilde{\omega}}_k[n_0+N-1] - \hat{\tilde{\omega}}_k[n_0+N]| < |\hat{\tilde{\omega}}_k[n_0+N-1] - \hat{\tilde{\omega}}_i[n_0+N]|$ for $i \neq k$. Finally, we obtain the initial estimates of ω_k, α_k and β_k by the criterion

$$[\hat{\omega}_k^{(0)}, \hat{\alpha}_k^{(0)}, \hat{\beta}_k^{(0)}] = \arg \max_{\omega \in \mathcal{F}_{k,\beta}} \boldsymbol{y}^H \boldsymbol{P}(\omega, \hat{\alpha}_k, \beta, \boldsymbol{\theta}_{k-1})\boldsymbol{y} \qquad k = 1, 2, \cdots, q \tag{13}$$



(c)



(d)

Fig. 2. (*Continued*) Performance comparison of the MMSE and alternating projection (AP) estimators for $q = 2, N = 21, a_1 = a_2 = 1.0$. (c) $f_2 = 0.4$. (d) $\alpha_2 = 0.02$ (experiment 2).

where $\mathcal{F}_k = \{\hat{\hat{\omega}}_k[n_0 + N] - 2\hat{\alpha}_k(n_0 + N) - 3\beta(n_0 + N)^2, \hat{\hat{\omega}}_k[n_0 + N] + \pi - 2\hat{\alpha}_k(n_0 + N) - 3\beta(n_0 + N)^2\}, \hat{\alpha}_k = 0.25(\hat{\hat{\omega}}_k[n_0 + N + 1] - \hat{\hat{\omega}}_k[n_0 + N - 1] - 12(n_0 + N)\beta), \boldsymbol{\theta}_k = [\hat{\omega}_1^{(0)}, \hat{\alpha}_1^{(0)}, \hat{\beta}_1^{(0)}, \cdots, \hat{\omega}_k^{(0)}, \hat{\alpha}_k^{(0)}, \hat{\beta}_k^{(0)}]$, and $\boldsymbol{\theta}_0 = \emptyset$ (empty set).

V. Computer Simulation Examples

To verify the performance of our MMSE estimator, we conducted two Monte Carlo experiments. In the first experiment, the data represented two linear chirp signals and were generated according to

$$\begin{aligned} y[n] = &(a_{10} + a_{11}n + a_{12}n^2)\exp\{j(\alpha_1 n^2 + \omega_1 n + \phi_1)\} \\ &+ (a_{20} + a_{21}n + a_{22}n^2)\exp\{j(\alpha_2 n^2 + \omega_2 n \\ &+ \phi_2)\} + w[n] \end{aligned}$$

where $a_{10} = 0.026\,72, a_{11} = 0.053\,43, a_{12} = -0.000\,534\,3, \alpha_1 = 0.000\,314\,2, \omega_1 = 0.062\,83, \phi_1 = 1.047, a_{20} = 0.0013, a_{21} = 0.001, a_{22} = -0.000\,234, \alpha_2 = 0.001\,014\,2, \omega_2 = 1.002\,83, \phi_2 = 0$, and $n = 0, 1, 2, \cdots, 100$. The results are shown in Fig. 1, where the plots (a) and (b) display the performance in estimating the initial frequency and chirp rate of the first signal, respectively. It is clear that the MMSE estimation performance is close to the CRLB when the SNR exceeds approximately -2 dB. Note that the CRLB's are obtained by computer matrix calculations. From the figure, we also see that the SNR threshold of our algorithm is lower than the thresholds of other schemes reported in the literature, such as the 3 dB in [5] for one linear chirp signal. In [5], the parameter estimator was based on the ML estimator implemented by the iterative Newton procedure. The Newton's method, however, is very sensitive to the initial search points, particularly when the data records are small. In our method, the sensitivity to initialization is smaller because the MMSE is a more robust estimator.

In the second experiment, we simulate a scenario of two linear chirps that are closely spaced in the time–frequency domain The data $y[n]$ were obtained from

$$\begin{aligned} y[n] = &\exp\left\{j2\pi\left(\frac{0.01}{2}n^2 + 0.2n\right)\right\} \\ &+ \exp\left\{j2\pi\left(\frac{0.02}{2}n^2 + 0.4n\right)\right\} + w[n] \end{aligned}$$

where $n = -10, -9, \cdots, 9, 10$. The results are shown in Fig. 2 and compared with the results of the alternating projection (AP) method. Clearly, for low SNR's, the MMSE algorithm significantly outperforms the AP method. This is so because the shape of the likelihood function is not a sharp peak in the $f - \alpha$ plane but a multitude of smaller peaks. This entails that the AP method becomes sensitive to initialization, which strongly deteriorates its performance.

In our method, however, this sensitivity is reduced by applying the concept of integration of multimodal functions with $M_2 = 2$ (see [6]). In addition, the MMSE algorithm implemented by the adaptive Gaussian quadrature numerical technique is much faster than the AP method.

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Detection of the Number of Signals in the Presence of White Noise in Decentralized Processing

Madhusudan Bhandary

***Abstract*— The problem of estimating the number of signals under a white noise model is solved when the data are received from many different places that are widely apart by using modified information criterion proposed by the author. This criterion involves the eigenvalues of the sample canonical correlation matrix. Some simulation results are also presented.**

***Index Terms*—Canonical correlation, information criterion, signal processing, white noise.**

I. INTRODUCTION

In the area of signal processing, q signals are observed at n different time points from different sources to p different sensors where $(q < p)$. However, due to atmospheric interference, the signals received by the sensors do not remain undistorted. Signals are affected by a noise factor. In this area, a model often used is that the observed signal vector is the sum of a random noise vector and a linear transform of a random signal vector. One of the important problems in this case is to estimate the number of signals transmitted. This problem is equivalent to estimating the multiplicity of smallest eigenvalue of the covariance matrix of the observation vector. Anderson [2], Krishnaiah [3], and Rao [5] considered the problem of testing the hypothesis of the multiplicity of the smallest eigenvalue of the covariance matrix. The problem of estimation of the number of signals by information theoretic criteria proposed by [1], [6], and [7] were condidered in [8], [10], and [11].

In this correspondence, we consider the problem of estimation of number of signals when the signals are coming from the same source but received by sensors placed at many different locations.

In Section II, we consider the model and assumptions for signal processing, and in Section III, we explain the model for our problem. Section IV describes the solution of the problem, and some simulation results are presented in Section V.

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II. MODEL AND ASSUMPTIONS

The following model [9] is used in signal processing:

$$\boldsymbol{x}(t) = A\boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{2.1}$$

where $\boldsymbol{x}(t) = (x_1(t), \cdots, x_p(t))$ is the $p \times 1$ observation vector at time t, $\boldsymbol{s}(t) = (s_1(t), \cdots, s_q(t))'$ is the $q \times 1$ vector of unknown random signals at time t, $\boldsymbol{n}(t) = (n_1(t), \cdots, n_p(t))'$ is the $p \times 1$ random noise vector at time t, and A is the $p \times q$ matrix of unknown parameters associated with the signals.

Under the assumption given in [9], we can say from model (2.1) that

$$\boldsymbol{x}(t) \sim N_p(\boldsymbol{0}, A\Psi A' + \Sigma_1).$$

The number of signals transmitted is q $(< p)$, which is the rank of $A\Psi A'$. Therefore, in this case, the estimation of the number of signals is equivalent to the estimation of the rank of $A\Psi A'$.

III. MODEL FOR THE PROBLEM

Our problem is described as follows: Suppose that q signals are going to p sensors in some location and to p sensors in some other location that is widely apart from the first location and to p sensors in some other location that is widely apart from the first two, and so on. Suppose in this way that there are k widely separated locations, each containing p sensors.

The statistical model in this case will be described by the k equations

$$\boldsymbol{x}_h(t) = A_h\boldsymbol{s}(t) + \boldsymbol{n}_h(t), \qquad h = 1, 2, \cdots, k \tag{3.1}$$

where $\boldsymbol{x}_h(t)$ is the $p \times 1$ observation vector at time t at the hth place. $\boldsymbol{s}(t)$ is the $q \times 1$ vector of unknown random signals, $\boldsymbol{n}_h(t)$ is the $p \times 1$ random noise vector at time t at the hth place. A_h is the $p \times q$ matrix of unknown parameters at the hth place $h = 1, 2, \cdots, k$.

Under the assumptions given in [9], it is trivial from model (3.1) that

$$\boldsymbol{x}_h \sim N_p(0, \Gamma_{hh} + \sigma_h^2 I_p)$$

where $\Gamma_{hh} = A_h \Psi A_h'$ is the nonnegative definite matrix of rank q $(< p)$, and

$$\begin{aligned} \text{Cov}\,(\boldsymbol{x}_h(t), \boldsymbol{x}_h'(t)) &= A_h \Psi A_{h'}' \\ &= \Gamma_{hh'}; \qquad h\ (\neq h') = 1, 2, \cdots, k \\ &= \text{matrix of rank } q\ (< p). \end{aligned} \tag{3.2}$$

We have n observations $\boldsymbol{x}_h(t_1), \boldsymbol{x}_h(t_2), \cdots, \boldsymbol{x}_h(t_n)$ at the hth place for $h = 1, 2, \cdots, k$, and on the basis of these observations, we